Bundles of C^*-Algebras

An Introduction to Dixmier-Douady Theory

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What is a bundle? - an analytic perspective

There are (at least) two different notions of bundle:

Let X be a locally compact Hausdorff space.

Definition

A C^*-algebra A is a **$C_0(X)$-algebra** if it comes equipped with a non-degenerate $*$-homomorphism

$$\theta\colon C_0(X) \to Z(M(A)) \ .$$

A is a **continuous $C_0(X)$-algebra** if the function $x \mapsto \|a(x)\|$ is continuous for all $a \in A$.

- given closed $Y \subset X$, let $A(Y) = A/C_0(X \setminus Y) \cdot A$,
- fibre of A at x is $A(x) := A(\{x\})$,
- let $a \in A$, denote the image of a in $A(x)$ by $a(x)$.

Examples of $C_0(X)$-algebras

- the **trivial $C_0(X)$-algebra** $C_0(X, B)$ with fibre B, ie. C_0-maps $f : X \to B$ for a C^*-algebra B, is a continuous $C_0(X)$-algebra,
- Let B be a unital C^*-algebra and consider

$$A = \{f \in C([-1, 1], B) \mid f(0) \in \mathbb{C}\, 1_B\} \ .$$

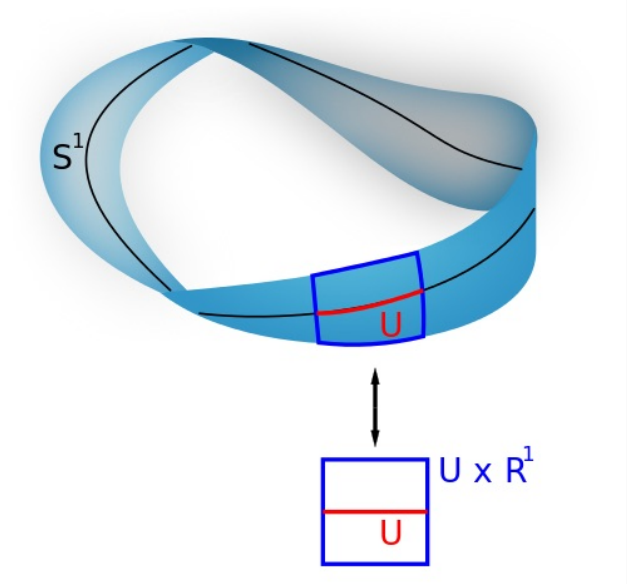
  This is also a continuous $C_0(X)$-algebra.
- Suppose we have a central extension of discrete countable amenable groups

$$1 \to N \to G \to H \to 1 \ .$$

  Then $C^*(G)$ is a continuous $C(\hat{N})$-algebra, where $\hat{N}$ is the Pontrjagin dual of N. The fibre over the trivial char. is $C^*(H)$.

What is a bundle? - a topological perspective

When topologists hear the word “bundle”, they think of something like this...



- locally a product (ie. locally trivial),
- but (possibly) non-trivial globally.

What is a bundle? - a topological perspective

Definition

A (locally trivial) **fibre bundle** $E \to X$ with **fibre** F consists of

- a topological space E, called the **total space**,
- a continuous map $\pi\colon E \to X$

with the following property:

- each point $x \in X$ has an open neighbourhood $U \subset X$ such that there exists a homeomorphism $\varphi_U\colon U \times F \to E|_U$ that makes

$$\begin{tikzcd} U \times F \arrow[rr, "\varphi_U"] \arrow[dr, "\mathrm{pr}_U"'] & & E|_U \arrow[dl, "\pi|_U"] \\ & U & \end{tikzcd}$$

  commute. The map φ_U is called a **trivialisation** over U.

What is a bundle? - a topological perspective

Let $E \to X$ be a fibre bundle with fibre F.

- Take an open cover $(U_i)_{i \in I}$ with trivialisations φ_i over U_i.
- The **transition maps** φ_{ij} over $U_{ij} = U_i \cap U_j$ are

$$\varphi_{ij} \colon U_{ij} \to \mathrm{Homeo}(F) \ .$$

  with $\varphi_{ij}(x)(f) = (\varphi_j^{-1} \circ \varphi_i)(x, f)$.
- Let $G \subset \mathrm{Homeo}(F)$. The bundle E has **structure group** G if we can find an open cover $(U_i)_{i \in I}$ such that all φ_{ij} factor through G.

Examples of fibre bundles

- vector bundles over a topological field k are fibre bundles with structure group $GL_n(k)$,
- hermitian vector bundles are fibre bundles with structure group $U(n)$
- manifold bundles with fibre a closed smooth manifold M are fibre bundles with structure group $\mathrm{Diff}(M)$,

And finally...

Definition

Let B be a C^*-algebra. A **bundle of C^*-algebras** $\mathcal{A} \to X$ with fibre B is a fibre bundle with structure group $\mathrm{Aut}(B)$ (equipped with the point-norm topology).

Homotopy classification of fibre bundles

Theorem

Let G be a topological group. There exists a topological space BG (called ***classifying space*** *of G) and a fibre bundle*

$$EG \to BG$$

with structure group G that has the following property:

For every compact Hausdorff space X and every fibre bundle $E \to X$ with structure group G there exists a continuous map $f : X \to BG$ (unique up to homotopy) such that

$$\begin{tikzcd} E \arrow[r] \arrow[d] & EG \arrow[d] \\ X \arrow[r, "f"] & BG \end{tikzcd}$$

is a pullback diagram.

Homotopy classification of fibre bundles

In other words...

Corollary

Let $\mathcal{B}un_G(X)$ be the set of isomorphism classes of fibre bundles with structure group G. There is a natural $1:1$-correspondence

$$\mathcal{B}un_G(X) \leftrightarrow [X, BG]$$

Examples of classifying spaces:

- $B\mathbb{Z} \simeq S^1$, hence $[X, B\mathbb{Z}] \cong [X, S^1] \cong H^1(X, \mathbb{Z})$.
- Hermitian line bundles have structure group $U(1)$ and

$$[X, BU(1)] \cong H^2(X, \mathbb{Z}) \ .$$

Back to $C_0(X)$-algebras

Let $\pi\colon \mathcal{A} \to X$ be a bundle of C^*-algebras with fibre B and consider

$$C_0(X, \mathcal{A}) = \{f\colon X \to \mathcal{A} \mid f \text{ is a } C_0\text{-map and } \pi \circ f = \mathrm{id}_X\}$$

This **section algebra** is a continuous $C_0(X)$-algebra.

Question

Given a $C_0(X)$-algebra A. When is it locally trivial?

1. Let $M_n = M_n(\mathbb{C})$ and consider

$$A = \{f \in C([-1,1], M_2) \mid f(0) \in \mathbb{C}\, 1_2\}$$

Is this isomorphic (as a $C([-1,1])$-algebra) to the trivial one?

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No! Fibre of A over 0 is $\cong \mathbb{C} \ncong M_2$.

Examples

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   $$1_2 \otimes e_{11} \in \mathbb{C}1_2 \otimes \mathbb{K}$$

   is a minimal projection in the fibre over 0 representing a generator in $K_0(A'(0))$.

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Observation

There is **no** continuous path in $\mathbf{hom}(\mathbb{K}, M_2 \otimes \mathbb{K})$ connecting $T \mapsto 1_2 \otimes T$ to an isomorphism.

Continuous $C_0(X)$-algebras with fibre $\mathbb{K}$

Reminder: $\mathbb{K} = \mathbb{K}(H)$ with H separable, ∞-dimensional.

Definition

Let A be a continuous $C_0(X)$-algebra with all fibres isomorphic to $\mathbb{K}$. It is said to satisfy **Fell's condition** if

$$\forall x \in X \, \exists \text{ closed nb. } V \text{ and } p \in A(V) \text{ s.th. } p(y) \text{ has rank } 1 \, \forall y \in V \, .$$

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Theorem (Fell)

Let A be a sep. continuous $C_0(X)$-algebra with fibre $\mathbb{K}$ over a locally compact, second countable space X of fin. dim. Then

$$A \text{ locally trivial} \quad \Leftrightarrow \quad A \text{ satisfies Fell's condition} \, .$$

Continuous $C_0(X)$-algebras with fibre $\mathbb{K}$

What about the homotopy classification?

The compacts $\mathbb{K}$ only have generalised inner automorphisms, ie.

$$\mathrm{Aut}(\mathbb{K}) \cong PU(H) := U(H)/U(1) \ .$$

Need to understand homotopy type of $BPU(H)$.

- $\pi_k(BPU(H)) \cong \pi_{k-1}(PU(H))$ (by classifying space theory),
- $\cdots \to \pi_k(U(H)) \to \pi_k(PU(H)) \to \pi_{k-1}(U(1)) \to \pi_{k-1}(U(H)) \to \ldots$
- but $U(H)$ contractible

$$\pi_k(B\mathrm{Aut}(\mathbb{K})) \cong \begin{cases} \mathbb{Z} & k = 3 \\ 0 & \text{else} \end{cases}$$

Continuous $C_0(X)$-algebras with fibre $\mathbb{K}$

Definition

Let G be a discrete group (abelian if $n \geq 2$). A space $K(G, n)$ with

$$\pi_k(K(G,n)) \cong \begin{cases} G & k = n \\ 0 & \text{else} \end{cases}$$

is called an **Eilenberg-Maclane space**. It is unique up to homotopy equivalence.

Theorem (Eilenberg-Steenrod)

Let X be a compact space. Then we have a natural isomorphism

$$[X, K(\mathbb{Z}, n)] \cong H^n(X, \mathbb{Z}) \ .$$

Continuous $C_0(X)$-algebras with fibre $\mathbb{K}$

Since $B\mathrm{Aut}(\mathbb{K}) \simeq K(\mathbb{Z}, 3)$ we have

Corollary (Dixmier-Douady)

We have a natural isomorphism:

$$\delta \colon \mathcal{B}un_{\mathbb{K}}(X) \to H^3(X, \mathbb{Z})$$

called the ***Dixmier-Douady class****. This isomorphism is multiplicative in the sense that*

$$\delta(A_1 \otimes_{C(X)} A_2) = \delta(A_1) + \delta(A_2) \ .$$

Let A^{op} be the continuous $C(X)$-algebra with reversed multiplication, then

$$\delta(A^{op}) = -\delta(A) \ .$$

Generalised Dixmier-Douady Theory - An Example

3 Remember our example from the beginning:

$$A = \{f \in C([-1,1], M_2) \mid f(0) \in \mathbb{C}\, 1_2\} \ .$$

Consider $A'' = A \otimes M_{2^\infty} \otimes \mathbb{K}$ with $M_{2^\infty} = M_2^{\otimes \infty}$.
Is this trivialisable?

Generalised Dixmier-Douady Theory - An Example

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$$A = \{f \in C([-1,1], M_2) \mid f(0) \in \mathbb{C}\, 1_2\} \ .$$

Consider $A'' = A \otimes M_{2^\infty} \otimes \mathbb{K}$ with $M_{2^\infty} = M_2^{\otimes \infty}$.
Is this trivialisable?

Yes! We can find a continuous path

$$\gamma \colon [0,1] \to \mathbf{hom}(M_{2^\infty} \otimes \mathbb{K}, M_2 \otimes M_{2^\infty} \otimes \mathbb{K})$$

with the properties

- $\gamma(0)(T) = 1_2 \otimes T$,
- $\gamma(t)$ is an isomorphism for each $t \in (0,1]$.

This gives us an isomorphism

$$C([-1,1], M_{2^\infty} \otimes \mathbb{K}) \to A''$$

by applying $\gamma(|t|)$ to $f(t)$ for $t \in [-1,1]$.

Generalised Dixmier-Douady Theory

This example works, since $D = M_{2^\infty}$ is strongly self-absorbing!

Definition (Toms-Winter)

A separable, unital C^*-algebra D is called **strongly self-absorbing** if $\exists$ an isomorphism $\psi\colon D \to D \otimes D$ and a path $u\colon [0,1) \to U(D \otimes D)$ with

$$\lim_{t\to 1} \|\psi(d) - u_t(d \otimes 1_D)u_t^*\| = 0$$

Some consequences of this definition:

- $\mathrm{Aut}(D)$ is contractible (and so is $B\mathrm{Aut}(D)$),
- $K_0(D)$ is a ring (and $K_1(D) = 0$ if D satisfies the UCT),

Generalised Dixmier-Douady Theory

Definition (Dadarlat-P.)

A continuous $C_0(X)$-algebra A with fibre $D \otimes \mathbb{K}$ for a strongly self-abs. C^*-algebra D satisfies the **generalised Fell condition** if

$$\forall x \in X \, \exists \text{ closed nb. } V \text{ and } p \in A(V) : [p(y)] \in GL_1(K_0(A(y))) \; \forall y \in V \, .$$

Generalised Dixmier-Douady Theory

Definition (Dadarlat-P.)

A continuous $C_0(X)$-algebra A with fibre $D \otimes \mathbb{K}$ for a strongly self-abs. C^*-algebra D satisfies the **generalised Fell condition** if

$$\forall x \in X \, \exists \text{ closed nb. } V \text{ and } p \in A(V) : [p(y)] \in GL_1(K_0(A(y))) \; \forall y \in V \; .$$

Theorem (Dadarlat-P.)

Let X be a locally compact space of finite covering dimension, let A be a separable $C_0(X)$-algebra as in the definition. Then

$$A \text{ locally trivial} \quad \Leftrightarrow \quad A \text{ satisfies the generalised Fell condition} \; .$$

C^*-algebra bundles with fibre $D \otimes \mathbb{K}$

What about the homotopy classification?

Theorem (Dadarlat-P.)

$\mathcal{B}un_{D\otimes\mathbb{K}}(X)$ is a group with respect to $\otimes_X$. In particular,

$$\mathcal{B}un_{M_{\mathbb{Q}}\otimes\mathbb{K}}(X) \cong H^1(X, \mathbb{Q}_+^\times) \oplus H^{odd,\geq 3}(X, \mathbb{Q}) \ ,$$
$$\mathcal{B}un_{M_{\mathbb{Q}}\otimes\mathcal{O}_\infty\otimes\mathbb{K}}(X) \cong H^1(X, \mathbb{Q}^\times) \oplus H^{odd,\geq 3}(X, \mathbb{Q})$$

If D is strongly self-absorbing, satisfies the UCT and $K_0(D) \neq 0$, then

$$D \otimes M_{\mathbb{Q}} \otimes \mathcal{O}_\infty \cong M_{\mathbb{Q}} \otimes \mathcal{O}_\infty \ .$$

This induces a homomorphism

$$\delta \colon \mathcal{B}un_{D\otimes\mathbb{K}}(X) \to H^1(X, \mathbb{Q}^\times) \oplus H^{\mathrm{odd},\geq 3}(X, \mathbb{Q}) \ .$$

(in particular: $\delta(A_1 \otimes_{C(X)} A_2) = \delta(A_1) + \delta(A_2)$).

C^*-algebra bundles with fibre $\mathcal{O}_\infty \otimes \mathbb{K}$

Reminder: $PU(H) \simeq K(\mathbb{Z}, 2)$

object	classifying space
hermitian line bundle L	$PU(H) \simeq BU(1)$
bundle of compact operators $\mathcal{A}$	$BPU(H) \simeq BBU(1)$

$U(1)$ is an abelian group $\quad\Rightarrow\quad$ $BU(1), BBU(1), \ldots$ exist

Observation: Line bundles form a subgroup in $GL_1(K^0(X))$.
Idea: Extend the above table to all of $GL_1(K^0(X))$!

object	classifying space
virtual line bundles	$GL_1(KU)$
?	$BGL_1(KU)$

$GL_1(KU)$ is an infinite loop space $\quad\Rightarrow\quad$ $BGL_1(KU)$ exists

C^*-algebra bundles with fibre $\mathcal{O}_\infty \otimes \mathbb{K}$

Theorem (Dadarlat-P.)

In general $\mathcal{B}un_{D\otimes\mathcal{O}_\infty\otimes\mathbb{K}}(X)$ is isomorphic to the first group of the cohomology theory associated to the unit spectrum of K-theory with coefficients $K_0(D)$. In particular, we have group isomorphisms:

$$\mathcal{B}un_{\mathcal{O}_\infty\otimes\mathbb{K}}(X) \cong [X, BGL_1(KU)] \ ,$$
$$\mathcal{B}un_{M_{n^\infty}\otimes\mathbb{K}}(X) \cong [X, BGL_1(KU[\tfrac{1}{n}])_+] \ .$$

Remarks:

- We also determined the homotopy type of $B\mathrm{Aut}(D \otimes \mathbb{K})$ for all strongly self-absorbing C^*-algebras D,
- The group $[X, BGL_1(KU)]$ and its variants can be determined via the Atiyah-Hirzebruch spectral sequence.

Equivariant higher Dixmier-Douady theory

Motivated by examples in twisted K-theory (jt. with David Evans) we might ask: Is there an equivariant generalisation of this?

Setting

- $G = \mathbb{T}$ (circle group),
- $D = \mathsf{End}(V)^{\otimes\infty}$ for $\mathbb{T}$-representation V,
- $\mathbb{K} = \mathbb{K}(H)$ for $H = \ell^2(\mathbb{Z}) \otimes H_0$, where $u_z(\delta_k) = z^k\,\delta_k$, $\mathsf{dim}(H_0) = \infty$ and H_0 separable.

Question: What is the homotopy type of $B\mathsf{Aut}_{\mathbb{T}}(D \otimes \mathbb{K})$?

Equivariant higher Dixmier-Douady theory (contd.)

Theorem (Evans-P.)

1. $\mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K})$ *is an infinite loop space with respect to* $\otimes$,
2. *The associated cohomology theory* $E^*_{D,\mathbb{T}}(X)$ *satisfies*

$$E^0_{D,\mathbb{T}}(X) = [X, \mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K})] \quad \textit{and} \quad E^1_{D,\mathbb{T}}(X) \cong [X, B\mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K})] \ ,$$

3. $E^k_{\mathbb{C},\mathbb{T}}(X) \cong H^k(X, \mathbb{Z}) \oplus H^{k+2}(X, \mathbb{Z})$ *and in particular*

$$E^1_{\mathbb{C},\mathbb{T}}(X) \cong H^3_{\mathbb{T}}(X, \mathbb{Z}) \cong Br_{\mathbb{T}}(X) \ ,$$

4. $E^k_{D,\mathbb{T}}(pt) \cong \pi_{-k}(\mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K}))$ *and we computed these groups.*

Understanding $E^1_{D,\mathbb{T}}(X)$

Theorem (Evans-P.)

$$E^1_{D,\mathbb{T}}(\mathbb{T}^n) \cong H^1(\mathbb{T}^n, G) \oplus H^3(\mathbb{T}^n, \mathsf{R}_{bdd}) \oplus \bigoplus_{k=2}^{\infty} H^{2k+1}(\mathbb{T}^n, \mathsf{R}^{0,\infty}_{bdd}) \ .$$

with $G = GL_1(K_0^{\mathbb{T}}(D))_+$.

The map $\theta\colon\ \mathsf{Aut}_{\mathbb{T}}(D \otimes \mathbb{K}) \to K_0^{\mathbb{T}}(D)$ given by

$$\theta(\alpha) = [\alpha(1 \otimes e)] \in K_0((D \otimes \mathbb{K})^{\mathbb{T}}) \cong K_0^{\mathbb{T}}(D)$$

factors through a homomorphism to $GL_1(K_0^{\mathbb{T}}(D))_+$ and is an isomorphism on π_0. It induces a natural transformation

$$\theta_*\colon E^1_{D,\mathbb{T}}(X) \to H^1(X, GL_1(K_0^{\mathbb{T}}(D))_+) \ .$$

Understanding $E^1_{D,\mathbb{T}}(X)$ for $D = \mathbb{C}$

Reminder: $E^1_{\mathbb{C},\mathbb{T}}(X) \cong \mathrm{Br}_{\mathbb{T}}(X) \cong H^1(X,\mathbb{Z}) \oplus H^3(X,\mathbb{Z})$

- For $D = \mathbb{C}$ we have $K_0^{\mathbb{T}}(\mathbb{C}) \cong \mathsf{Rep}(\mathbb{T}) \cong \mathbb{Z}[t, t^{-1}]$.
- Therefore $GL_1(K_0^{\mathbb{T}}(\mathbb{C})) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ and $GL_1(K_0^{\mathbb{T}}(\mathbb{C}))_+ \cong \mathbb{Z}$.

Hence,

$$\theta_* \colon E^1_{\mathbb{C},\mathbb{T}}(X) \to H^1(X,\mathbb{Z}) \ .$$

This is the Phillips-Raeburn obstruction:

$$\mathbb{K}\text{-bundle } \mathcal{A} \to X \ \rightsquigarrow \ \mathbb{K}^{\mathbb{T}}\text{-bundle } \mathcal{A}^{\mathbb{T}} \to X \ \rightsquigarrow \ \mathbb{Z}\text{-bundle } \widehat{\mathcal{A}^{\mathbb{T}}} \to X \ .$$

where $\mathbb{K}^{\mathbb{T}} \cong C_0(\mathbb{Z}) \otimes \mathbb{K}(H_0)$ and $\widehat{\mathcal{A}^{\mathbb{T}}} \to X$ takes the spectrum fibrewise.

$$F \colon E^1_{\mathbb{C},\mathbb{T}}(X) \to H^3(X,\mathbb{Z})$$

gives the Dixmier-Douady class after forgetting the $\mathbb{T}$-action.

What is next?

- Fell bundles with unit fibre D give interesting examples. Can we compute the Dixmier-Douady classes?
- Is there a version of Chern-Weil theory to compute the generalised rational Dixmier-Douady classes?
- Can we use our results to say something about bundles of C^*-algebras with fibre $\mathcal{O}_n$? (work of Taro Sogabe)
- There are other interesting group actions on strongly self-absorbing C^*-algebras.

  Can we say anything about $\mathrm{Aut}_G(D \otimes \mathbb{K})$ then?

The End

Thank you!