GAUGE THEORY WITH FINITE GAUGE GROUPS

1. Basics about principal bundles and classifying spaces

In the first lecture we discussed how Witten's work on gauge theories [4] revealed a deep connection between certain 3-dimensional topological quantum field theories and knot invariants. The problem with the heuristic argument was the definition of the path integral on the space of connection forms on principal $SU(2)$-bundles $P \to X$. One way to circumvent this problem is to consider a *finite* gauge group G instead of $SU(2)$ as has been done by Freed and Quinn in [2] (see also [1]). In this way, all technicalities involving measures on infinite dimensional spaces disappear and one is left with a nice natural toy model of a TQFT, which already reveals many of its features. In this note, which summarizes material from [2, 1], we will discuss the 2-dimensional case.

Definition 1.1. Let G be a topological group. A *principal G-bundle* on a space X is a space P together with a projection map $\pi\colon P \to X$, such that G acts on P from the right, the map π is equivariant (where X is equipped with the trivial action of G) and X has a covering by open sets U, such that there exist G-equivariant homeomorphisms $\phi_U\colon \pi^{-1}(U) \to U \times G$ fitting into the following commuting diagram:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi_U} & U \times G \\ {\scriptstyle\pi}\downarrow & \swarrow{\scriptstyle \mathrm{pr}_U} & \\ U & & \end{array}$$

If there exists a subordinate partition of unity for the trivializing cover, the principal bundle is called *numerable*. Note that this is always the case, if X is paracompact. Two principal G-bundles P, P' are *isomorphic*, if there exists a continuous G-equivariant map $\kappa\colon P \to P'$ covering the identity on X (exercise: Any such map is automatically a homeomorphism.). We will denote the set of isomorphism classes of numerable principal G-bundles over X by $\mathrm{Prin}_G(X)$.

If $f\colon Y \to X$ is a continuous map and $\pi\colon P \to X$ is a (numerable) principal G-bundle, then the bundle

$$f^*P = \{(y,p) \in Y \times P \mid f(y) = \pi(p)\}$$

is a (numerable) principal G-bundle over Y with respect to the canonical projection map $(y,p) \mapsto y$. This bundle f^*P is called the *pullback* of P to Y. If $f, h\colon Y \to X$ are homotopic, the bundles f^*P and h^*P are isomorphic. For any topological group G there is a space BG and a principal G-bundle $EG \to BG$, such that for any topological space X the map

$$[X, BG] \to \mathrm{Prin}_G(X) \quad ; \quad f \mapsto f^*(EG)$$

is a bijection. The space BG is called the *classifying space* of G, $EG \to BG$ is the *universal principal G-bundle*. BG is unique up to homotopy equivalence. If EG is a contractible space with free G-action, then EG/G is a model for BG. The construction of BG is functorial in the sense that any continuous group homomorphism $\alpha\colon H \to G$ induces a map $B\alpha\colon BH \to BG$ classifying the principal G-bundle

$$EH \times_\alpha G = EH \times G/\sim$$

over BH, where the equivalence relation is generated by $(eh, g) \sim (e, \alpha(h)g)$ and the projection maps (e, g) to $[e] \in BH$. A nice introduction to principal G-bundles can be found in [3, Chapter 5 and Chapter 7].

Example 1.2. The universal cover of S^1 is $\mathbb{R}$, which is contractible and carries a free action of $\mathbb{Z}$ such that $\mathbb{R}/\mathbb{Z} \simeq S^1$. Therefore S^1 has the homotopy type of $B\mathbb{Z}$.

Example 1.3. If Σ is a closed, smooth, oriented 2-dimensional manifold of genus $g \geq 1$, then its universal cover $\widetilde{\Sigma}$ is contractible and the fundamental group $\pi = \pi_1(\Sigma)$ acts freely on $\widetilde{\Sigma}$ via deck transformations, such that $\Sigma \simeq \widetilde{\Sigma}/\pi$. Thus, Σ is a model for $B\pi$ in this case.

Example 1.4. If G is a discrete group. Then $EG \to BG$ agrees with the universal cover of BG. If H is another discrete group, then any continuous map $f\colon BG \to BH$ is homotopic to $B\alpha$ with $\alpha = \pi_1(f)\colon G \to H$. To see this, note that f can be lifted to a map $\widetilde{f}\colon EG \to EH$, such that the following diagram commutes

$$\begin{array}{ccc} EG & \xrightarrow{\widetilde{f}} & EH \\ \downarrow & & \downarrow \\ BG & \xrightarrow{f} & BH \end{array}$$

and $\widetilde{f}$ satisfies $\widetilde{f}(eg) = \widetilde{f}(e)\,\alpha(g)$. This induces an isomorphism of principal H-bundles

$$EG \times_\alpha H \to f^*EH \quad ; \quad (e, h) \mapsto \widetilde{f}(e)\,h\ .$$

Thus, the bundle classified by $B\alpha$ is isomorphic to the bundle classified by f. This implies that f and $B\alpha$ are homotopic. Suppose X is a connected, locally path-connected and semi-locally simply connected space (e.g. a connected manifold). Then the universal cover $\widetilde{X}$ exists and a similar argument as above may be used to show that a continuous map $g\colon X \to BG$ is homotopic to $f \circ B\alpha$ where $\alpha = \pi_1(g)$ and $f\colon X \to B\pi$ is the map classifying the universal cover. This implies that

$$\mathrm{Hom}(\pi_1(X), G) \to [X, BG] \quad ; \quad \alpha \mapsto f \circ B\alpha$$

is surjective. The bundle classified by $f \circ B\alpha$ is $\widetilde{X} \times_\alpha G$. Given a *pointed* principal G-bundle $\pi\colon P \to X$ with basepoint $p_0 \in P$ over the basepoint $x_0 = \pi(p_0) \in X$ we can obtain a corresponding $\alpha \in \mathrm{Hom}(\pi_1(X), G)$ as follows: Take a closed curve $\gamma\colon S^1 \to X$ with $\gamma(1) = x_0$. This represents an element $h \in \pi_1(X)$. Its lift to P at p_0 corresponds to a curve from p_0 to $p_0\,\alpha(h^{-1})$. Since the action of G on P is free, this determines α completely. α is called the *monodromy* of the bundle. Choosing a different basepoint $p_0' \in P$ over x_0 yields $\alpha' = \mathrm{Ad}_g \circ \alpha \in \mathrm{Hom}(\pi_1(X), G)$ for some $g \in G$. Checking the details we obtain a bijection

$$\mathrm{Hom}(\pi_1(X), G)/G \xrightarrow{\sim} \mathrm{Prin}_G(X)\ .$$

2. Gauge theory and TQFT

Now let G be a finite group. From the considerations about classifying spaces above, we get a bijection

$$\mathrm{Prin}_G(S^1) \cong [B\mathbb{Z}, BG] \cong \mathrm{Hom}(\mathbb{Z}, G)/\!\sim\; = G/\!\sim$$

where the equivalence relations are generated by conjugation in G, i.e. $\alpha \sim \mathrm{Ad}_g \circ \alpha$ and $g \sim hgh^{-1}$ respectively. This is a finite set, therefore for a fixed field k the vector space $\mathrm{Map}(\mathrm{Prin}_G(S^1), k)$ is finite-dimensional. Let X and Y be spaces for which $\mathrm{Prin}_G(X)$ and $\mathrm{Prin}_G(Y)$ are finite, then we have

$$\begin{aligned}\mathrm{Map}(\mathrm{Prin}_G(X \sqcup Y), k) &\cong \mathrm{Map}(\mathrm{Prin}_G(X) \sqcup \mathrm{Prin}_G(Y), k)\\ &\cong \mathrm{Map}(\mathrm{Prin}_G(X), k) \otimes \mathrm{Map}(\mathrm{Prin}_G(Y), k)\ .\end{aligned}$$

There is a unique map $\emptyset \to \emptyset$, which turns $\emptyset$ into a principal G-bundle for any G – the empty bundle (note that the action of G is given by the canonical identification $\emptyset \times G \to \emptyset$). This quite degenerate case leads to the conclusion

$$\mathrm{Map}(\mathrm{Prin}_G(\emptyset), k) \cong k\ .$$

It encourages us to set $Z(\Sigma) = \mathrm{Map}(\mathrm{Prin}_G(\Sigma), k)$ for a closed, smooth, oriented 1-dimensional manifold Σ.

Since we only consider discrete groups, any principal G-bundle P over a smooth manifold M can be equipped with a smooth structure such that the projection is a local diffeomorphism. Since this structure is unique, we will not distinguish between smooth and continuous bundles any further. Note that if $\Sigma \subset M$ is a compact, smooth, oriented submanifold of the compact, smooth, oriented manifold M, then restricting a principal G-bundle $P \to M$ to Σ yields a principal G-bundle $P|_\Sigma \to \Sigma$. Since isomorphic principal bundles give isomorphic restrictions, this induces a map

$$\mathrm{Prin}_G(M) \to \mathrm{Prin}_G(\Sigma) \quad ; \quad p \mapsto p|_\Sigma\ .$$

If $P \to M$ is a principal G-bundle, then we denote by $\mathrm{Aut}(P)$ the G-equivariant homeomorphisms $P \to P$ that cover the identity of M. $\mathrm{Aut}(P)$ is in bijection with continuous maps $P \to G$ that are equivariant with respect to the adjoint action of G on itself. Since these are locally constant maps, there is no difference in considering continuous or smooth automorphisms. Isomorphic principal bundles have isomorphic automorphism groups, therefore $\#\mathrm{Aut}(p)$ is well-defined for $p \in \mathrm{Prin}_G(M)$.

Let M be a surface with oriented boundary components, in-boundary Σ_1 and out-boundary Σ_2 and let $C_M(p_1, p_2) = \{p \in \mathrm{Prin}_G(M) \mid p|_{\Sigma_1} = p_1 \text{ and } p|_{\Sigma_2} = p_2\}$. Whenever we have such a cobordism, we can define a linear map[1]

$$Z(M)\colon Z(\Sigma_1) \to Z(\Sigma_2) \quad ; \quad Z(M)(f)(p_2) = \sum_{p_1 \in \mathrm{Prin}_G(\Sigma_1)} \sum_{p \in C_M(p_1,p_2)} f(p_1) \frac{\#\mathrm{Aut}(p_2)}{\#\mathrm{Aut}(p)}\ .$$

[1]We use the slightly different space $C_M(p_1, p_2)$ compared to $C_M(Q)$ that appears in [2, 1], since we think this is the more natural choice. Therefore, our measure needs a different normalization.

This expression is a direct translation of the path integral from physics to our discrete situation. The role of the measure is played by $\mu_{p_1,p_2}(p) = \frac{\#\mathrm{Aut}(p_2)}{\#\mathrm{Aut}(p)}$ for $p \in C_M(p_1, p_2)$. To see how μ_{p_1,p_2} behaves with respect to gluing, let $Q \to \Sigma_2$ be a pricnipal G-bundle and let $\mathcal{C}_M(p_1, Q)$ be the category with objects given by pairs (P, θ), where $P \to M$ is a principal G-bundle with $[P|_{\Sigma_1}] = p_1$ and $\theta\colon\ P|_{\Sigma_2} \to Q$ is an isomorphism. A morphism $(P, \theta) \to (P', \theta')$ is an isomorphism of principal G-bundles $\varphi\colon P \to P'$, such that $\theta' \circ \varphi|_{\Sigma_2} = \theta$. Observe that an element in $\mathrm{Aut}(P, \theta)$ has to be the identity on the component of Σ_2. We denote the isomorphism classes of objects in this category by $C_M(p_1, Q)$. We have the following gluing lemma:

Lemma 2.1. *Let M_i be two smooth, oriented manifolds with oriented boundary components, such that Σ is an out-boundary of M_1 and an in-boundary of M_2. Let $P_i \to M_i$ be principal G-bundles over M_i, $Q \to \Sigma$ be a principal G-bundle over Σ and let $\theta_i\colon\ P_i|_\Sigma \to Q$ be isomorphisms. Let $\iota_k\colon M_k \to M_1 \coprod_\Sigma M_2$ be the inclusion maps. Then there exists a principal G-bundle $P \to M_1 \coprod_\Sigma M_2$ such that $\iota_k^* P$ is canonically isomorphic to P_k.*

Proof. Let $P = P_1 \coprod P_2/\sim$, where the equivalence relation identifies the point $\theta_1(p)$ with $\theta_2(p)$. Since the isomorphisms are by definition equivariant, P turns out to be a principal G-bundle with the desired properties. □

Let us denote the result of the gluing by $P_1 \coprod_{\Sigma_2} P_2$. This construction descends to the isomorphism classes of $\mathcal{C}_M(p_1, Q) \times \mathcal{C}_M(Q, p_3)$ and yields a map

$$\Phi_Q\colon C_M(p_1, Q) \times C_M(Q, p_3) \to C_M(p_1, p_3) \quad ; \quad ([P_1, \theta_1], [P_2, \theta_2]) \mapsto \left[P_1 \coprod_{\Sigma_2} P_2\right] .$$

The restriction of principal G-bundles $P \to M_1 \coprod_{\Sigma_2} M_2 = M$ to the submanifolds M_1 and M_2 yields

$$R\colon C_M(p_1, p_3) \to \coprod_{q \in \mathrm{Prin}_G(\Sigma_2)} C_{M_1}(p_1, q) \times C_{M_2}(q, p_3) .$$

For each element $q \in \mathrm{Prin}_G(\Sigma_2)$ choose a representative Q and denote by I the set of all these. Then we have a commutative diagram

$$\begin{array}{ccc} & & \coprod_{Q \in I} C_{M_1}(p_1, Q) \times C_{M_2}(Q, p_3) \\ & \swarrow^{\Phi} & \downarrow \\ C_M(p_1, p_3) & \xrightarrow{R} & \coprod_{q \in \mathrm{Prin}_G(\Sigma_2)} C_{M_1}(p_1, q) \times C_{M_2}(q, p_3) \end{array}$$

where the arrow on the right hand side is induced by $\varphi_{p_1,Q}\colon C_{M_1}(p_1, Q) \to C_{M_1}(p_1, q)$; $[P, \theta] \mapsto [P]$ and the corresponding map φ_{Q,p_3}. As a direct analogue to $\mu_{p_1,q}$, we have a similar measures on $C_{M_1}(p_1, Q)$ and $C_{M_1}(Q, p_3)$ given by $\widehat{\mu}_{p_1,Q}([P, \theta]) = \frac{\#\mathrm{Aut}(Q)}{\#\mathrm{Aut}(P,\theta)}$ and $\widehat{\mu}_{Q,p_3}([P', \theta']) = \frac{\#\mathrm{Aut}(p_3)}{\#\mathrm{Aut}(P',\theta')}$ respectively. Observe that we have an exact sequence

$$1 \to \mathrm{Aut}(P, \theta) \times \mathrm{Aut}(P', \theta') \to \mathrm{Aut}\left(P \coprod_{\Sigma_2} P'\right) \xrightarrow{\theta_*} \mathrm{Aut}(Q) ,$$

where the last map is induced by restricting the automorphism to $P|_{\Sigma_2}$ and conjugating the result with $\theta\colon P|_{\Sigma_2} \to Q$. Exactness is due to the fact that an element $\kappa \in \mathrm{Aut}(P)$ lies in $\mathrm{Aut}(P,\theta)$ precisely if it is the identity on P restricted to the component of Σ_2. The map to $\mathrm{Aut}(Q)$ is not surjective. To see what its image is, note that $\mathrm{Aut}(Q)$ acts on $C_{M_1}(p_1,Q) \times C_{M_2}(Q,p_3)$ by precomposition, i.e. for $\kappa \in \mathrm{Aut}(Q)$ and $((P,\theta),(P',\theta')) \in C_{M_1}(p_1,Q) \times C_{M_2}(Q,p_3)$, the action is given by

$$\kappa \cdot ((P,\theta),(P',\theta')) = ((P,\kappa\circ\theta),(P',\kappa\circ\theta')) \ .$$

In particular, it maps $\Phi_Q^{-1}(P\coprod_{\Sigma_2} P')$ into itself and is transitive there. Moreover, (P,θ) is isomorphic to $(P,\kappa\circ\theta)$, say via α, exactly when $\kappa = \theta^{-1}\circ\alpha|_{\Sigma_2}\circ\theta$, i.e. when κ lies in the image of the last map discussed above. Therefore the stabilizer of the action of $\mathrm{Aut}(Q)$ on $\Phi_Q^{-1}(P\coprod_{\Sigma_2} P')$ coincides with the image of θ_*. Counting the orders of the groups we arrive at

$$\#\mathrm{Aut}\left(P\coprod_{\Sigma_2} P'\right) = \#\mathrm{Aut}(P,\theta)\,\#\mathrm{Aut}(P',\theta')\,\frac{\#\mathrm{Aut}(Q)}{\#\Phi_Q^{-1}(P\coprod_{\Sigma_2} P')} \ .$$

Thus, for a fixed Q and $\widetilde{P} = P\coprod_{\Sigma_2} P'$ glued over Q we have

$$\begin{aligned}\Phi_{Q*}(\widehat{\mu}_{p_1,Q}\times\widehat{\mu}_{Q,p_3})(\widetilde{P}) &= (\widehat{\mu}_{p_1,Q}\times\widehat{\mu}_{Q,p_3})(\Phi_Q^{-1}(\widetilde{P})) = \sum_{\theta,\theta'} \widehat{\mu}_{p_1,Q}(P,\theta)\,\widehat{\mu}_{Q,p_3}(P',\theta') \\ &= \sum_{\theta,\theta'} \frac{\mu_{p_1,p_3}(\widetilde{P})\,\#\mathrm{Aut}(Q)^2}{\#\Phi_Q^{-1}(P\coprod_{\Sigma_2} P')} = \mu_{p_1,p_3}(\widetilde{P})\,\#\mathrm{Aut}(Q)^2 \ .\end{aligned}$$

A similar argument using the exact sequence $1 \to \mathrm{Aut}(P,\theta) \to \mathrm{Aut}(P) \to \mathrm{Aut}(Q)$ shows that $(\varphi_{p_1,Q*}\times\varphi_{Q,p_3*})(\widehat{\mu}_{p_1,Q}\times\widehat{\mu}_{Q,p_3}) = \#\mathrm{Aut}(Q)^2\,(\mu_{p_1,q}\times\mu_{q,p_3})$. Therefore

$$R_*(\mu_{p_1,p_3})(P,P') = \frac{1}{\#\mathrm{Aut}(Q)^2}\,(R_*\circ\Phi_{Q*})(\widehat{\mu}_{p_1,Q}\times\widehat{\mu}_{Q,p_3})(P,P') = (\mu_{p_1,q}\times\mu_{q,p_3})(P,P')$$

Thus, $R_*(\mu_{p_1,p_3}) = \sum_{q\in\mathrm{Prin}_G(\Sigma_2)} \mu_{p_1,q}\times\mu_{q,p_3}$, i.e. R is well-behaved with respect to restricting glued bundles.

Lemma 2.2. *Let M_1 be a cobordism between Σ_1 and Σ_2, let M_2 be a cobordism between Σ_2 and Σ_3. Then, with the definition of $Z(M_i)$ as above, we have*

$$Z\left(M_1\coprod_{\Sigma_2} M_2\right) = Z(M_2)\circ Z(M_1) \ .$$

Proof. By computation:

$$\begin{aligned}(Z(M_2)\circ Z(M_1)(f))(p_3) &= \sum_{p_1\in \mathrm{Prin}_G(\Sigma_1)} \sum_{(p,p')\in \coprod_q C_{M_1}(p_1,q)\times C_{M_2}(q,p_3)} f(p_1)(\mu_{p_1,q}\times \mu_{q,p_3})(p,p') \\ &= \sum_{p_1\in \mathrm{Prin}_G(\Sigma_1)} \sum_{(p,p')\in \coprod_q C_{M_1}(p_1,q)\times C_{M_2}(q,p_3)} f(p_1) R_*\mu_{p_1,p_3}(p,p') \\ &= \sum_{p_1\in \mathrm{Prin}_G(\Sigma_1)} \sum_{\widetilde{p}\in C_M(p_1,p_3)} f(p_1)\mu_{p_1,p_3}(\widetilde{p}) = Z\left(M_1 \coprod_{\Sigma_2} M_2\right)(f)(p_3)\ .\end{aligned}$$

□

If $M = \Sigma\times I$ is a cylinder, $C_{\Sigma\times I}(p_1,p_2)$ is empty whenever $p_1\neq p_2$ and contains the single element $[P_1\times I]$ if $p_1=[P_1]=p_2$. Since $\mathrm{Aut}(P_1\times I)\cong \mathrm{Aut}(P_1)$. Thus, $Z(\Sigma\times I)=\mathrm{id}$. Thus, we have proven:

Theorem 2.3. *Z as above defines a $2D$-TQFT.*

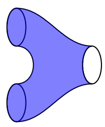
2.1. **Gauge theory and the classification of $2D$-TQFTs.** By the classification theorem for 2-dimensional TQFTs Z should correspond to a Frobenius algebra given by $Z(S^1)$. To see the algebraic structure, note first that

$$\mathrm{Map}(\mathrm{Prin}_G(S^1),k) = \mathrm{Map}(G,k)^G$$

is the set of functions on G, which are invariant under the adjoint action of G on itself. It is an exercise to see that these corresponds bijectively to elements in the center[2] $Z(kG)$ of the group ring kG, with respect to the identification

(1) $$\mathrm{Map}(G,k)\xrightarrow{\sim} kG \quad ; \quad f\mapsto \sum_{g\in G} f(g)\,g$$

Consider the pair of pants cobordism M_μ shown below.

$M_\mu =$ 

This surface is homeomorphic to a disc with two smaller open discs removed. The fundamental group of this space is isomorphic to $\mathbb{F}_2=\langle a,b\rangle$, the free group in two generators a and b. Therefore principal G-bundles over M_μ are in $1:1$-correspondence with elements in $\mathrm{Hom}(\mathbb{F}_2,G)/G$. An element φ in $\mathrm{Hom}(\mathbb{F}_2,G)$ is completely fixed by $\varphi(a)$ and $\varphi(b)$. If φ classifies a principal G-bundle P over M_μ, then its restriction to the two circles Σ_1 and Σ_2 on the left hand side of the above picture correspond to the (conjugacy classes of the) group elements $\varphi(a)$ and $\varphi(b)$, whereas the restriction to the circle on the right hand side is

[2]Unfortunately, the letter Z is used for both construction in the literature.

classified by the conjugacy class of $\varphi(ab) = \varphi(a)\,\varphi(b)$. Thus, the map $Z(M_\mu)$ is given by

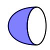
$$Z(M_\mu)\colon \mathrm{Map}(G,k)^G \otimes \mathrm{Map}(G,k)^G \to \mathrm{Map}(G,k)^G$$

$$Z(M_\mu)(f_1 \otimes f_2)(g) = \sum_{h_1 h_2 = g} f_1(h_1)\, f_2(h_2)\ .$$

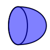
If we identify $\mathrm{Map}(G,k)^G$ with $Z(kG)$ via (1) we end up with

$$Z(M_\mu)\colon Z(kG) \otimes Z(kG) \to Z(kG) \quad ; \quad \sum_{h_1 \in G} \lambda_{h_1} h_1 \otimes \sum_{h_1 \in G} \mu_{h_2} h_2 \mapsto \sum_{h_1,h_2 \in G} \lambda_{h_1} \mu_{h_2} h_1 h_2\ ,$$

which is just the usual multiplication map in the group ring. For the unit cobordism, i.e.

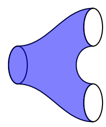
$M_u =$ 

we have $Z(M_u)(\lambda)(g) = \lambda\, \delta_{g,e}$, where $e \in G$ is the identity in the group and $\delta_{g,e}$ is 1 if $g = e$ and 0 else, where we consider $Z(M_u)$ as a map $k \to \mathrm{Map}(G,k)^G$. This is due to the fact, that any principal G-bundle over a disc is isomorphic to the trivial bundle. The corresponding linear map $k \to Z(kG)$ sends λ to $\lambda\, e$. Therefore $Z(S^1)$ is isomorphic to $Z(kG)$ *as a unital algebra*. The linear form $\tau = Z(M_\tau)\colon \mathrm{Map}(G,k)^G \to k$ obtained from

$M_\tau =$ 

sends a map f to $f(e)$ for the same reason as above. When considered on the center of the group ring $\tau\colon Z(kG) \to k$ maps $\sum_{g \in G} \lambda_g\, g$ to λ_e. This determines the Frobenius algebra structure on $Z(S^1) = Z(kG)$ completely.

2.2. **Gauge theory and representation theory.** In general the comultiplication, i.e. $\delta = Z(M_\delta)$ with

$M_\delta =$  ,

will send $1 \in Z(S^1)$ to some element $\sum_{i \in I} e_i \otimes f_i \in Z(S^1) \otimes Z(S^1)$. To find e_i and f_i we will stick to the case $k = \mathbb{C}$ from now on and need some basic notions from the representation theory of finite groups:

Definition 2.4. Let V be an n-dimensional complex vector space. A group homomorphism $\rho\colon G \to \mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$ is called a (complex, finite dimensional) *representation* of the group G on the vector space V. A representation is called *irreducible* if the only elements of $\mathrm{End}(V) \cong M_n(\mathbb{C})$ that commute with $\rho(g)$ for all $g \in G$ are multiples of the identity. The *character* of a representation ρ is defined to be

$$\chi_\rho\colon G \to \mathbb{C} \quad ; \quad \chi_\rho(g) = \mathrm{tr}(\rho(g))\ .$$

Two representations $\rho_1\colon G \to U(V)$ and $\rho_2\colon G \to U(V')$ are *equivalent*, if there exists an isomorphism $u\colon V \to V'$ such that $\mathrm{Ad}_u \circ \rho_1 = \rho_2$.

By definition the character of a representation is invariant under the conjugation action and therefore provides an element of $\mathrm{Map}(G,\mathbb{C})^G \cong Z(\mathbb{C}G)$. The character only depends on the equivalence class of the representation. Now we have:

Theorem 2.5. *Let G be a finite group and denote by Δ the set of equivalence classes of irreducible finite dimensional representations. Then Δ is finite and the ring $Z(\mathbb{C}G)$ decomposes into a direct sum of copies of $\mathbb{C}$ labeled by the elements of Δ. If $e_\rho \in Z(\mathbb{C}G)$ is a projection onto one of these direct summands, then*

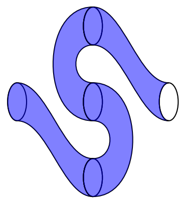
$$e_\rho = \sum_{g\in G} \frac{\dim(V_\rho)}{\#G} \chi_\rho(g)\, g \ ,$$

where $\chi_\rho \colon G \to \mathbb{C}$ is the character of $\rho \in \Delta$. Moreover, the χ_ρ satisfy

$$\sum_{\rho\in\Delta} \chi_\rho(g)\chi_\rho(h^{-1}) = \begin{cases} \#C_G(g) & \text{if } g \text{ and } h \text{ are conjugate} \\ 0 & \text{else} \end{cases} ,$$

where $C_G(g)$ is the centralizer of the element $g \in G$.

Consider the element $x = (\#G)^{-1} \sum_{\rho\in\Delta} \chi_\rho \otimes \chi_\rho \in \mathrm{Map}(G,\mathbb{C})^G \otimes \mathrm{Map}(G,\mathbb{C})^G$ corresponding to $(\#G)^{-1} \sum_{\rho\in\Delta} \sum_{g,h\in G} \chi_\rho(g)\chi_\rho(h)\, g \otimes h \in Z(\mathbb{C}G) \otimes Z(\mathbb{C}G)$. Let $y = \sum_{g\in G} \lambda_g g \in Z(\mathbb{C}G)$. If we evaluate $Z(M_s)(y)$ for

$M_s =$ 

we end up with

$$Z(M_s)(y) = (\#G)^{-1} \sum_{\rho\in\Delta, g,h\in G} \chi_\rho(g)\, \chi_\rho(h^{-1})\, \lambda_h\, g$$

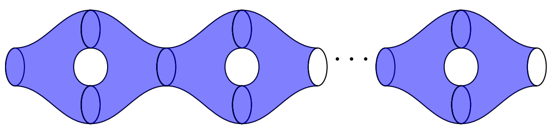
If g and h are conjugate, then $\lambda_g = \lambda_h$ because $y \in Z(\mathbb{C}G)$. Moreover, the centralizer is the stabilizer with respect to the adjoint action of the group on itself and the orbits of the adjoint action are the conjugacy classes $[g]$. This implies $\#C_G(g) \cdot \#[g] = \#G$. Therefore the above evaluates to

$$Z(M_s)(y) = (\#G)^{-1} \sum_{g\in G} (\#C_G(g) \cdot \#[g])\, \lambda_g\, g = \sum_{g\in G} \lambda_g\, g = y$$

and similarly for the other snake relation. Since these linear maps determine the copairing and also the coproduct, we have $\delta(y) = Z(M_\delta)(y) = xy$ with x as above. Observe that we can rewrite x as follows

$$x = (\#G)^{-1} \sum_{\rho\in\Delta} \left(\frac{\#G}{\dim(V_\rho)} \right)^2 e_\rho \otimes e_\rho \ .$$

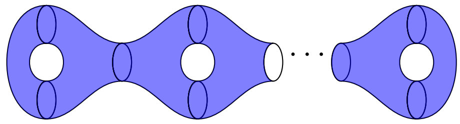
Now let M_g be the following cobordism consisting of g multiplications and comultiplications:

$M_g =$  .

Then we have

$$Z(M_g)(e_\tau) = (\#G)^{-1}\left(\frac{\#G}{\dim(V_\tau)}\right)^{2g} e_\tau$$

Therefore for a closed connected surface

$N_g =$ 

we obtain

$$Z(N_g)(1) = \sum_{\rho\in\Delta} \tau\left(Z(M_g)(e_\rho)\right) = (\#G)^{-1}\sum_{\rho\in\Delta}\left(\frac{\#G}{\dim(V_\rho)}\right)^{2g}\tau(e_\rho) = \sum_{\rho\in\Delta}\left(\frac{\#G}{\dim(V_\rho)}\right)^{2g-2}.$$

The second way to obtain the value of $Z(N_g)$ at 1 is to directly evaluate the path integral/sum above. Let $\Gamma_g = \pi_1(N_g)$. As we observed in the first section, the set of isomorphism classes of principal G-bundles is in bijection with $\mathrm{Hom}(\Gamma_g, G)/G$. Let $P \to N_g$ be a principal G-bundle with monodromy $\alpha\colon \Gamma_g \to G$ and let $\varphi\colon P \to P$ be an automorphism. P is isomorphic to $\widetilde{N}_g \times_\alpha G$, where $\widetilde{N}_g$ is the universal cover of N_g. φ corresponds to a map $\widetilde{\varphi}\colon \widetilde{N}_g \to G$, such that

$$\varphi\colon \widetilde{N}_g \times_\alpha G \to \widetilde{N}_g \times_\alpha G \quad ; \quad (\widetilde{x}, g) \mapsto (\widetilde{x}, \widetilde{\varphi}(\widetilde{x})g) .$$

Because G is discrete and $\widetilde{N}_g$ is connected, $\widetilde{\varphi}$ is constant. Let $h \in G$ be its value. h has to satisfy $\mathrm{Ad}_h \circ \alpha = \alpha$. Therefore $\mathrm{Aut}(P)$ is in bijection with the stabilizer of α with respect the adjoint action of G on $\mathrm{Hom}(\Gamma_g, G)$. Thus, we have

$$\begin{aligned} Z(N_g)(1) &= \sum_{p\in \mathrm{Prin}_G(N_g)} \frac{1}{\#\mathrm{Aut}(p)} = \sum_{[\alpha]\in \mathrm{Hom}(\Gamma_g,G)/G} \frac{1}{\# C_{\mathrm{Hom}(\Gamma_g,G)}(\alpha)} \\ &= \sum_{\alpha\in \mathrm{Hom}(\Gamma_g,G)} \frac{1}{\#[\alpha]\,\# C_{\mathrm{Hom}(\Gamma_g,G)}(\alpha)} = \sum_{\alpha\in \mathrm{Hom}(\Gamma_g,G)} \frac{1}{\#G} = \frac{\#\mathrm{Hom}(\Gamma_g, G)}{\#G} \end{aligned}$$

From the gluing formula of the TQFT we therefore arrive at the following relationship between homomorphisms of the fundamental group of a surface into a finite group G and the representation theory of G:

$$\#\mathrm{Hom}(\Gamma_g, G) = (\#G)^{2g-1}\sum_{\rho\in\Delta} \dim(V_\rho)^{2-2g}$$

This is a non-obvious result, which can also be deduced using group theory. As we have seen, it also follows from the gluing rule of the 2-dimensional TQFT.

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