Algebraic Topology
– An Overview –

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What is Mathematics about?

Quantitative Questions

- What is the value of the integral $\int_0^1 x^2 \, dx$?
- What is the volume of a ball in $\mathbb{R}^3$ of radius r?
- What is the ratio of a circle's circumference to its diameter?

- $\int_0^1 x^2 \, dx = \frac{1}{3}$.
- $\mathrm{Vol(Ball)} = \frac{4}{3}\pi r^3$, where r is its radius.
- If c is the circumference and d the diameter, we have $\frac{c}{d} = \pi$.

However, . . .

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However, . . .

Mathematics is not (only) about numbers!

What is Mathematics about?

Qualitative Questions

- Are two given vector spaces V and W isomorphic to each other?
- Does the sequence $a_n = \sum_{k=0}^{n} q^k$ for $q \in \mathbb{R}$ converge?

Answer	$\rightsquigarrow$	**Concept**
• $V \cong W$ if and only if $\dim(V) = \dim(V)$.	$\rightsquigarrow$	dimension
• If $\lvert q \rvert < 1$ we have $\lim_{n\to\infty} a_n = \frac{1}{1-q}$.	$\rightsquigarrow$	geometric series

Mathematics provides a language...

- to make precise qualitative statements, which are either true or false,
- to prove or disprove these statements.

What is Topology?

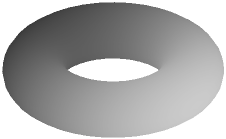
Topology provides a precise language to talk about ...

- continuity, continuous maps,
- closeness, neighbourhoods of points
- continuous deformations

... in the most general setting possible!

Example: Can we continuously deform ...

into



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What is Topology?

Answer to the example question

It is possible to deform the surface of a coffee cup into the one of a doughnut. The picture below shows a "sketch" of the proof.

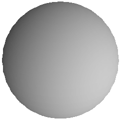
Figure: The deformation of a coffee cup into a doughnut.

What is Topology?

Figure: Animation of the continuous deformation

What is Topology?

Another example: Can we continuously deform . . .



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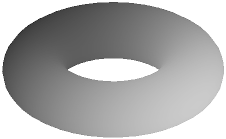

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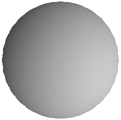
What is Topology?

Another example: Can we continuously deform . . .



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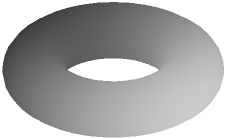

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Here the answer seems to be: “No!”. But how do we prove it?

This is were **algebraic topology** comes into play.

What is Algebraic Topology?

Topological invariants (of surfaces)

Suppose that we had a construction that produces a number $\chi(X) \in \mathbb{Z}$ for any surface X with the property that whenever X can be continuously deformed into Y, we have

$$\chi(X) = \chi(Y) \ .$$

Such a number is an example of a **topological invariant**.

Observation:

If we can compute that $\chi(X) \neq \chi(Y)$ for given X and Y, then X can not be continuously deformed into Y.

What is Algebraic Topology?

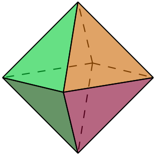
Algebraic topology provides tools to...

- translate difficult problems in topology into algebraic problems that are often easier to solve,
- study topological invariants in the most general setting.

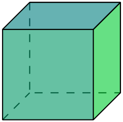
Remarks:

- There are topological invariants that are not numbers, but other algebraic structures, like groups.
- In this case a continuous deformation produces not the same group, but an isomorphic one.

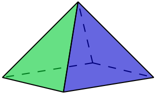
The Euler characteristic



Octahedron
$V = 6$
$E = 12$
$F = 8$



Cube
$V = 8$
$E = 12$
$F = 6$



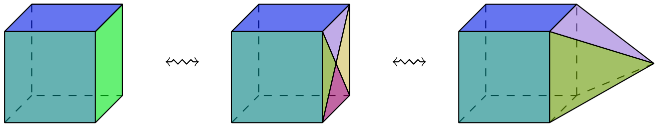
Pyramid
$V = 5$
$E = 8$
$F = 5$

$$\chi(P) = V - E + F = 2$$

in all the examples above.

The Euler characteristic

What do we mean by continuous deformations in this particular context?

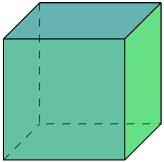


Deformed Cube
$V = 9$
$E = 16$
$F = 9$

Observation: The Euler characteristic remains constant under this move.

The Euler characteristic

Example: Can we use these moves to deform. . .



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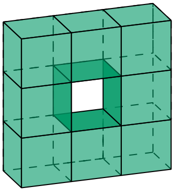

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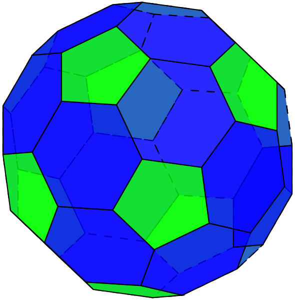
Cube:
$V = 8$
$E = 12$
$F = 6$
$\chi(\text{Cube}) = 2$

Doughnut:
$V = 32$
$E = 64$
$F = 32$
$\chi(\text{Doughnut}) = 0$

Footballs and chemistry



A football is composed of...

- $F_6 = 20$ hexagons (faces bounded by 6 edges)
- $F_5 = 12$ pentagons (faces bounded by 5 edges)

Question: Does it have to look like that?
Are there other possible values for F_5 and F_6?

Footballs and chemistry

More precise question

Let X be a surface, which satisfies the following conditions:

- It is composed of
  - F_5 pentagons,
  - F_6 hexagons

  glued along the edges (no other faces).
- It is "shaped like a sphere".

What are the possible values for F_5 and F_6?

Condition about the shape of X gives:

- $\chi(X) = 2$,
- three edges meet at each vertex.

The maths slide

- V - number of vertices of X,
- E - number of edges,
- F - number of faces.

The following equations hold:

$$
\begin{aligned}
V - E + F &= 2 \\
F &= F_5 + F_6 \\
2E &= 5F_5 + 6F_6 \\
3V = 2E &= 5F_5 + 6F_6
\end{aligned}
$$

Expressing the first equation in terms of F_5 and F_6:

$$
\begin{aligned}
\frac{5F_5 + 6F_6}{3} - \frac{5F_5 + 6F_6}{2} + F_5 + F_6 &= 2 \\
10F_5 + 12F_6 - (15F_5 + 18F_6) + 6F_5 + 6F_6 &= 12 \\
F_5 &= 12
\end{aligned}
$$

Footballs and chemistry

Question:

Let X be a surface, which satisfies the following conditions:

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What are the possible values for F_5 and F_6?

Answer – The 12-pentagon theorem

- These conditions imply that $F_5 = 12$.
- We must have $2F_6 = V - 20$, where V is the number of vertices.

Find the error. . .

What is wrong with the following signs?

Find the error...

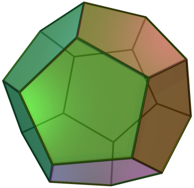
What is wrong with the following signs?

There is no football with $F_5 = 0$ as depicted in the signs!

This is excluded by the 12-pentagon theorem.

Footballs and chemistry

In fact, there is a surface X with $F_6 = 0$, which satisfies the conditions:



Footballs and chemistry

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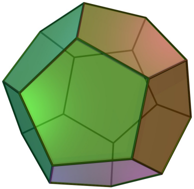
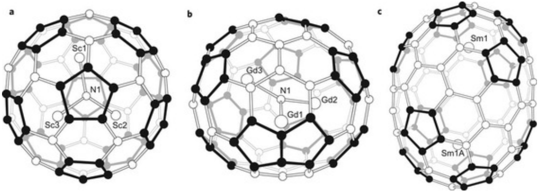
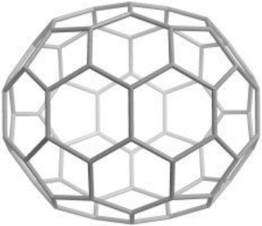


The 12-pentagon theorem has applications in chemistry: No matter how large a fullerene is, its atomic structure has to contain 12 pentagons.



Finally, some rugby . . .



The fullerene C_{70} has 70 atoms. Hence, we get

- $V = 70$,
- $2F_6 = V - 20$, therefore $F_6 = 25$,
- $F_5 = 12$.

It is also called the rugby ball.

A puzzle involving a picture frame

Question

Can you hang a picture frame with a string looped around two nails in such a way that if you remove either one of the two nails, the picture falls down?

Figure: This is not a solution, but it illustrates the setup.

A puzzle involving a picture frame

Simplifications

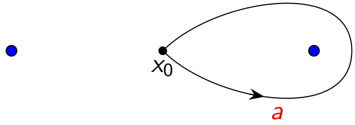
- Forget about the picture frame, just tie the ends of the string together to get a loop.
- Fix a base point x_0 of the loop, where it starts and ends.
- Think about the nails as holes obstructing the movement of the string.
- Think of the rope like an infinitely elastic rubber band, which can be extended and shrunk to arbitrary (finite) lengths.

Rephrased Question

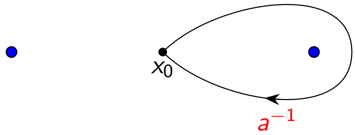
Can we find a loop starting at x_0 around the two holes, which can not be contracted to the point x_0, but becomes contractible whenever one of the two holes is filled in?

The maths

Suppose x_0 is the point in the middle between the two holes. A loop that runs around the nail on the right hand side counterclockwise looks like this:



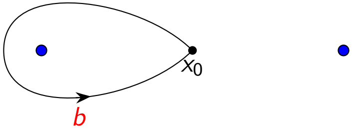
We will call this loop a. If we run around the same nail clockwise we get:



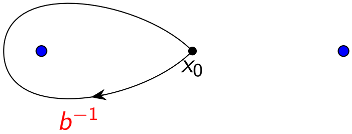
This loop will be called a^{-1}. This notation will be explained later.

The maths

Of course we can also run around the hole on the left hand side counterclockwise. This gives the following loop which we will call b:

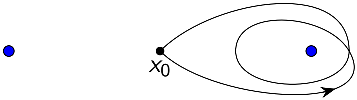


and its clockwise counterpart will be called b^{-1} again:



The maths

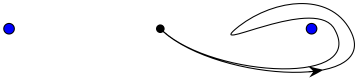
Using this notation we can also compose loops. The picture below shows the loop $a^2 = aa$.



If we call the contractible loop e, then we see that we have

$$aa^{-1} = a^{-1}a = e \quad , \quad bb^{-1} = b^{-1}b = e \; .$$

as illustrated in the following picture:



The maths

If we are only interested in (oriented) loops starting at x_0 up to deformation, then those configurations correspond precisely to the "words" that can be formed using the letters a, b, a^{-1}, b^{-1}.

We also allow the "empty word", which we will denote by e, since it corresponds to the contractible loop.

Denote the set of all those words by $\mathbb{F}_2$.

The maths

The set $\mathbb{F}_2$ has very interesting properties:

- We can compose elements of $\mathbb{F}_2$ by concatenation, where we use the rules

$$aa^{-1} = a^{-1}a = bb^{-1} = b^{-1}b = e \ .$$

- This concatenation of words is associative.
- The letter e acts as a neutral element, i.e. for any word $w \in \mathbb{F}_2$ we have

$$we = ew = w \ .$$

- Every word $w \in \mathbb{F}_2$ has an inverse word w^{-1} with the property

$$ww^{-1} = w^{-1}w = e \ .$$

You might have seen these properties before. They show that the set $\mathbb{F}_2$ together with the concatenation operation is a **group**. This group is called the **free group on the two generators a and b**.

The maths

We can now solve our original question using **algebra**!

Filling in the hole on the right hand side corresponds to forgetting all a's or a^{-1}'s in a word, i.e. setting those letters equal to e. Similarly for the hole on the left hand side and b, respectively b^{-1}.

Algebra Question

Which words $w \in \mathbb{F}_2$ with $w \neq e$ become equal to e if we set all a and a^{-1} equal to e and leave b and b^{-1} untouched, and also if we set all b and b^{-1} equal to e without changing a and a^{-1}.

This question can be solved much easier! Here are some solutions:

- $w = aba^{-1}b^{-1}$.
- $w = a^2ba^{-2}b^{-1}$.
- $w = aba^{-1}b^{-1}aba^{-1}b^{-1}aba^{-1}b^{-1}$

A solution

From any such word we get a solution to the original question. For example, if we pick $w = b^{-1}a^{-1}ba$, we end up with the string looping around the two nails as shown below:

Figure: A solution to the puzzle.