A Non-commutative Model for Higher Twisted K-Theory

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Idea of K-Theory:

Capture the algebraic structure of vector bundles over topological spaces.

- X - topological space (compact and Hausdorff)
- $\mathcal{V}ect_{\mathbb{C}}(X)$ - category of finite dimensional $\mathbb{C}$-vector bundles over X

Definition

The 0th K-group of X is defined to be the abelian group

$$K^0(X) = \mathrm{Gr}(\mathrm{obj}(\mathcal{V}ect_{\mathbb{C}}(X)), \oplus)$$

where $\mathrm{Gr}(M)$ denotes the Grothendieck group of the monoid M.

Operator K-Theory

Definition

The 0th K-group of a unital C^*-algebra A is defined as

$$K_0(A) = \mathrm{Gr}(\mathrm{obj}(\mathcal{P}roj\mathcal{M}od(A)), \oplus)$$

where $\mathcal{P}roj\mathcal{M}od(A)$ denotes the category of projective Hilbert A-modules.

Properties:

- $X \mapsto K^i(X)$ exists for all $i \in \mathbb{Z}$ and is a generalised cohomology theory
- $A \mapsto K_i(A)$ exists for all $i \in \mathbb{Z}$.
- $K^*(X)$ is a graded ring via $\otimes$.
- Bott periodicity

$$K^i(X) \cong K^{i+2}(X) \qquad , \qquad K_i(A) \cong K_{i+2}(A)$$

Twisted K-Theory

Theorem (Serre-Swan)

Let X be a compact Hausdorff space. Then

$$K^*(X) \cong K_*(C(X)) \cong K_*(C(X, \mathbb{K})) \ .$$

Definition

Let $\pi \colon \mathcal{K} \to X$ be a (locally trivial) bundle of compact operators. The twisted K-theory of X with twist $\mathcal{K}$ is defined as

$$K^*_{\mathcal{K}}(X) = K_*(C(X, \mathcal{K}))$$

where $C(X, \mathcal{K}) = \{f \colon X \to \mathcal{K} \text{ continuous} \mid \pi \circ f = \mathrm{id}_X\}$.

Loop Groups and the Verlinde Ring

Theorem (Freed, Hopkins, Teleman)
Let G be a simply connected compact Lie group and let LG be the free loop group on G. Then we have

$$K^{\dim(G)}_{G,\mathcal{K}(k)}(G^{adj}) \cong Rep_k(LG)\ .$$

Other Applications

- index theory on non-spin^c manifolds (Stolz, Rosenberg, ...)
- modular invariants (Evans, Gannon)

Classification of Twists

Theorem (Dixmier-Douady)
X paracompact Hausdorff space. Then

$$\mathcal{B}un_{\mathbb{K}}(X) \cong H^3(X, \mathbb{Z}) \ .$$

- locally trivial $\mathbb{K}$-bundle $\mathcal{K} \to X$ over X
  $\rightsquigarrow$ transition maps $\varphi_{ij} \colon U_{ij} = U_i \cap U_j \to PU(H)$.
- pulling back the $U(1)$-bundle $U(H) \to PU(H)$ via φ_{ij} gives $L_{ij} \to U_{ij}$
- multiplication in $U(H)$ gives $\mu_{ijk} \colon L_{ij} \otimes L_{jk} \to L_{ik}$ + associativity
- choose sections $\sigma_{ij} \colon U_{ij} \to L_{ij}$, have $\mu_{ijk}(\sigma_{ij} \otimes \sigma_{jk}) = \omega_{ijk} \cdot \sigma_{ik}$
  for continuous maps $\omega_{ijk} \colon U_{ijk} \to U(1)$
- The family $(\omega_{ijk})_{i,j,k}$ represents element in $H^2(X, U(1)) \cong H^3(X, \mathbb{Z})$

What are Higher Twists?

- $H^3(X,\mathbb{Z})$-twists of K-Theory $\subseteq$ bundles of units in K-Theory.

object	classifying space
hermitian line bundle L	$PU(H) \simeq BU(1)$
bundle of compact operators $\mathcal{A}$	$BPU(H) \simeq BBU(1)$

$BU(1)$ is an infinite loop space
(tensor product is "associative enough") $\Rightarrow BBU(1)$ exists

- higher twists of K-Theory $=$ bundles of units in K-Theory.

object	classifying space
units $=$ virtual line bundles	$GL_1(KU)$
?	$BGL_1(KU)$

$GL_1(KU)$ is an infinite loop space $\Rightarrow BGL_1(KU)$ exists

A Non-commutative Model for Higher Twists

Dixmier-Douady: We have $\mathcal{B}un_{\mathbb{K}}(X) \cong H^3(X, \mathbb{Z}) \cong [X, BBU(1)]$.

Question

Is there a C^*-algebra A, such that

$$\mathcal{B}un_A(X) \cong [X, BGL_1(KU)] \ ?$$

Definition (Toms-Winter)

A separable unital C^*-algebra D is called strongly self-absorbing if there exists an iso. $\psi\colon D \to D \otimes D$ and a cont. path $u\colon [0,1) \to U(D \otimes D)$ s.th. for all $d \in D$

$$\lim_{t \to 1} \|\psi(d) - u_t\,(d \otimes 1)\,u_t^*\| = 0 \ .$$

Strongly Self-Absorbing C^*-algebras

Examples

- infinite tensor product of $M_n(\mathbb{C})$
- more generally: infinite UHF-algebras, like the universal UHF-algebra
- the Cuntz algebras $\mathcal{O}_\infty$ and $\mathcal{O}_2$
- the Jiang-Su algebra $\mathcal{Z}$

Some Properties:

- $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ if and only if A is simple, separable, unital and nuclear
- A simple, separable and nuclear, then $A \cong A \otimes \mathcal{O}_\infty$ if and only if A is purely infinite
- **Conjecture (Toms-Winter):** A simple, separable and nuclear, then $A \cong A \otimes \mathcal{Z}$ if and only if $\dim_{\text{nuc}}(A) < \infty$

A Non-commutative Model for Higher Twists (contd.)

Let D be a unital C^*-algebra,
let X be a compact metrizable space

$$\begin{aligned} D \text{ self-absorbing} \quad &\rightsquigarrow \quad \mathcal{B}un_{D\otimes\mathbb{K}}(X) \text{ is a semigroup w.r.t. } \otimes \\ D \text{ strongly self-absorbing} \quad &\rightsquigarrow \quad \mathcal{B}un_{D\otimes\mathbb{K}}(X) \text{ is a monoid w.r.t. } \otimes \end{aligned}$$

In fact, we have

Theorem (Dadarlat, P.)

$(\mathcal{B}un_{D\otimes\mathbb{K}}(X), \otimes)$ *is a group, which is the first group of a generalised cohomology theory.* $BAut(D \otimes \mathbb{K})$ *is an infinite loop space.*

A Non-commutative Model for Higher Twists (contd.)

Examples

- $D = M_{\mathbb{Q}}$ the universal UHF-algebra

$$\mathcal{B}un_{M_{\mathbb{Q}} \otimes \mathbb{K}}(X) \cong H^1(X, \mathbb{Q}_+^\times) \oplus \bigoplus_{n=1}^{\infty} H^{2n+1}(X, \mathbb{Q})$$

- $D = M_{\mathbb{Q}} \otimes \mathcal{O}_\infty$

$$\mathcal{B}un_{M_{\mathbb{Q}} \otimes \mathcal{O}_\infty \otimes \mathbb{K}}(X) \cong H^1(X, \mathbb{Q}^\times) \oplus \bigoplus_{n=1}^{\infty} H^{2n+1}(X, \mathbb{Q})$$

- $D = \mathcal{O}_\infty$

$$\mathcal{B}un_{\mathcal{O}_\infty \otimes \mathbb{K}}(X) \cong [X, BGL_1(KU)]$$

- for general strongly self-absorbing D

$$\mathcal{B}un_{D \otimes \mathbb{K}}(X) \cong [X, BGL_1(KU^D)]$$

Higher Twisted K-Theory

Our results give the following interpretation of higher twisted K-theory:

- $\mathcal{A}$ – locally trivial bundle with fiber $\mathcal{O}_\infty \otimes \mathbb{K}$
- Define higher twisted K-theory of X via

$$K^*_{\mathcal{A}}(X) = K_*(C(X, \mathcal{A}))$$

Theorem (P.)
$K^*_{\mathcal{A}}(X)$ agrees with higher twisted K-theory as defined by Ando, Blumberg, Gepner, Hopkins and Rezk.

$K^{\mathcal{A}}_*(X) := KK(C(X, \mathcal{A}), \mathbb{C})$ also agrees with their version of twisted K-homology for finite CW-complexes X.

Thank you!

Torsion elements in $\mathcal{B}un_{D\otimes\mathbb{K}}(X)$

What about bundles with fiber D instead of $D \otimes \mathbb{K}$?

Theorem (Dadarlat, P.)
Let D be a strongly self-absorbing C^*-algebra. Then the classifying space $BAut(D)$ is contractible, in particular $\mathcal{B}un_D(X)$ is trivial.

However, there is a surjection

$$\coprod_{n\in\mathbb{N}} \mathcal{B}un_{M_n(D)}(X) \to \mathrm{Tor}\,(\mathcal{B}un_{D\otimes\mathbb{K}}(X))$$

via fibrewise stabilisation (i.e. $\otimes\mathbb{K}$). For $D = \mathbb{C}$ this reduces to a theorem of Serre.

Construction of $\mathcal{O}_\infty$

- H - separable Hilbert space $\rightsquigarrow$ Fock space $\mathcal{F}(H) = \bigotimes_{n=0}^{\infty} H^{\otimes n}$
- each $\xi \in H$ yields $s_\xi : \mathcal{F}(H) \to \mathcal{F}(H)$ with $s_\xi(\eta) = \xi \otimes \eta$
- $\mathcal{O}_\infty$ is the closed unital $*$-subalgebra of $B(\mathcal{F}(H))$ generated by the s_ξ's and their adjoints s_ξ^*

Construction of a $\mathbb{K}$-bundle

Examples

- (V, q) - complex vector space with quadratic form q
  $\rightsquigarrow$ Clifford algebra $\mathbb{C}\ell(V, q)$
- (M, g) Riemannian manifold with metric g
  $\rightsquigarrow$ Clifford bundle $\mathbb{C}\ell(TM)$ over M
- if M is even dimensional, $\mathbb{C}\ell(TM)$ is a bundle of matrix algebras

Open Questions

- Is there an equivariant version of this description?
  If yes, how does it relate to the Freed-Hopkins-Teleman theorem?
- Is there such a nice description of the generalized rational Dixmier-Douady classes as for the case $D = \mathbb{C}$?
- Topologists have studied the algebraic K-theory of topological K-theory $K(ku)$, which comes with a natural map

  $$BGL_1(KU) \to K(ku)$$

  Does the above description extend to one of $K(ku)$? Does this produce classes in that theory?