Loops, Groups and Twists

The role of K-theory in mathematical physics

Ulrich Pennig[1]

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Pure Maths Colloquium, Southampton

[1]Cardiff University
PennigU@cardiff.ac.uk

From Topology to Algebra

Cohomology theories have the following form: For $n \in \mathbb{Z}$ we have

$$\begin{aligned} E^n \colon Top &\to \mathcal{C} \\ X &\mapsto E^n(X) \\ (f\colon Y \to X) &\mapsto (f^* \colon E^n(X) \to E^n(Y)) \end{aligned}$$

where $\mathcal{C}$ is the category of abelian groups.
They are homotopy invariant: If $f, g\colon Y \to X$ are homotopic, then $f^* = g^*$.

Figure 1: A homotopy equivalence

Topological K-theory

Idea of K-theory:

Capture the topology of a space by studying vector bundles over it.

- X - topological space (compact and Hausdorff)
- $\mathcal{V}_{\mathbb{C}}(X)$ - isomorphism classes of fin. dim. $\mathbb{C}$-vector bundles over X

Note that $\mathcal{V}_{\mathbb{C}}(X)$ is a monoid with respect to $\oplus$.

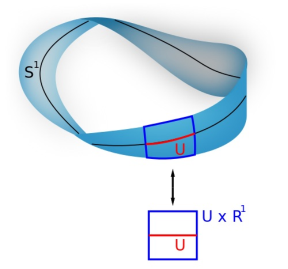

Figure 2: The Möbius strip as a vector bundle

Topological K-theory

If $f\colon Y \to X$ is a continuous map, then we have

$$f^* : \mathcal{V}_{\mathbb{C}}(X) \to \mathcal{V}_{\mathbb{C}}(Y)$$

defined via pullback of vector bundles.

Lemma
If $f, g\colon Y \to X$ are homotopic, then $f^* = g^*$.

Turn $\mathcal{V}_{\mathbb{C}}(X)$ into an abelian group!

$$(E, F) \sim (E', F') \quad \Leftrightarrow \quad \exists G \in \mathcal{V}_{\mathbb{C}}(X) \text{ with } E \oplus F' \oplus G = E' \oplus F \oplus G$$

Definition
The 0th K-group of the compact Hausdorff space X is defined to be

$$K^0(X) = \mathcal{V}_{\mathbb{C}}(X) \times \mathcal{V}_{\mathbb{C}}(X)/\sim$$

Denote $[E, F]$ by $E - F$.

Topological K-theory

Further properties:

- $X \mapsto K^n(X)$ exists for all $n \in \mathbb{Z}$
- Bott periodicity:
$$K^i(X) \cong K^{i+2}(X)$$
- $K^1(X) \cong [X, U_\infty]$, where $U_\infty = \mathrm{colim}_n U(n)$.
- $K^*(X)$ is a graded ring via $\otimes$.

K^* **is a cohomology theory!**

Example
For the $2n$-sphere S^{2n} with $n \geq 1$ we have
$$K^0(S^{2n}) \cong \mathbb{Z}[\gamma]/(\gamma^2) \qquad \text{and} \qquad K^1(S^{2n}) = 0\ .$$

Definition

Denote by $\mathcal{B}(H)$ the bounded operators on a Hilbert space. An algebra A is called a C^*-algebra if it is a norm-closed $*$-subalgebra of $\mathcal{B}(H)$.

Example

- $C_0(X)$ for a locally compact Hausdorff space,
- $\mathbb{K} = \mathbb{K}(H)$, i.e. the compact operators on a Hilbert space H,
- $C_0(X, \mathcal{A})$, where $\mathcal{A} \to X$ is a locally trivial bundle of C^*-algebras,
- $M_n^{\otimes\infty}$, i.e. the infinite tensor product of $M_n(\mathbb{C})$.

Operator K-Theory

Idea: Vector bundle over X $\equiv$ fin. gen. proj. module over $C(X)$

- A - unital C^*-algebra
- $\mathcal{P}_{\mathbb{C}}(A)$ - iso. classes of fin. gen. projective right Hilbert A-modules

Note that $\mathcal{P}_{\mathbb{C}}(A)$ is a monoid with respect to $\oplus$.

Definition

The 0th K-group of a unital C^*-algebra A is defined as

$$K_0(A) = \mathcal{P}_{\mathbb{C}}(A) \times \mathcal{P}_{\mathbb{C}}(A)/\sim$$

where $\sim$ is a similar equivalence relation as before.

Properties:

- $A \mapsto K_n(A)$ exists for all $n \in \mathbb{Z}$.
- Bott periodicity: $K_i(A) \cong K_{i+2}(A)$

From Algebra to Topology

Theorem (Adams, Atiyah)

If $\mathbb{R}^k$ is a division algebra, then $k \in \{1, 2, 4, 8\}$.

Lemma

If $\mathbb{R}^k$ for $k > 1$ is a division algebra, then k is even.

Proof.

- Assume $k = 2n + 1$.
- $\mu\colon \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ induces $\mu\colon S^{2n} \times S^{2n} \to S^{2n}$.
- Have ring homomorphism

$$\mu^*\colon K^0(S^{2n}) \to K^0(S^{2n}) \otimes K^0(S^{2n})$$

- which we can identify with

$$\mu^*\colon \mathbb{Z}[\gamma]/(\gamma^2) \to \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$$

From Algebra to Topology

Proof.

- $\mu^*\colon \mathbb{Z}[\gamma]/(\gamma^2) \to \mathbb{Z}[\alpha,\beta]/(\alpha^2,\beta^2)$
- Let $\iota\colon S^{2n} \to S^{2n} \times S^{2n}$ be given by $\iota(x) = (x, 1)$.
- $\iota^*\colon \mathbb{Z}[\alpha,\beta]/(\alpha^2,\beta^2) \to \mathbb{Z}[\gamma]/(\gamma^2)$ satisfies $\iota^*(\alpha) = \gamma$ and $\iota^*(\beta) = 0$.
- Have $\mu^*(\gamma) = r\alpha + s\beta + t\alpha\beta$ for some integers r, s, t.
- Since $\iota^* \circ \mu^* = (\mu \circ \iota)^* = id^*$ we get

  $$r\gamma = \iota^*(\mu^*(\gamma)) = \gamma$$

  and hence $r = 1$. Similarly, $s = 1$.
- But then

  $$0 = \mu^*(\gamma^2) = \mu^*(\gamma)^2 = (\alpha + \beta + t\alpha\beta)^2 = 2\alpha\beta \neq 0$$

  which is a contradiction. □

Applications of K-theory

- Classification of topological insulators via real and complex K-theory (Bourne-Kellendonk-Rennie, Kitaev, Loring-Hastings, ...)
- D-brane charges in string theory via twisted K-theory (Bouwknegt-Mathai, Bouwknegt-Carey-Mathai-Murray-Stevenson,...)
- Topological T-duality (Bouwknegt-Evslin-Mathai, Bunke-Schick, Mathai-Rosenberg, ...)
- Modular tensor categories from loop groups, CFTs and TQFTs (Evans-Gannon, Freed-Hopkins-Teleman, Meinrenken, ...)

What are loop groups?

- G - compact Lie group (will assume simply connected later)
- $LG = C^\infty(\mathbb{T}, G)$ - group of smooth loops in G

Even though LG are infinite-dimensional groups, they still have a feasible representation theory!

- If G is simply connected, then LG has a universal central $\mathbb{T}$-extension

$$1 \to \mathbb{T} \to \widetilde{LG} \to LG \to 1$$

(Denote the central copy of $\mathbb{T}$ by $\mathbb{T}_c$)

- The action of $\mathbb{T}$ on LG by rotating the circle lifts to $\widetilde{LG}$ giving rise to the group $\mathbb{T} \ltimes \widetilde{LG}$. (Denote the rotation circle by $\mathbb{T}_r$).

Loop Groups and the Verlinde Ring

Definition

A representation $\rho\colon \mathbb{T}_r \ltimes \widetilde{LG} \to U(H)$ on a Hilbert space H is called a positive energy representation if $\rho|_{\mathbb{T}_r} : \mathbb{T}_r \to U(H)$ decomposes H into a sum of finite-dimensional subspaces on each of which $\mathbb{T}_r$ acts by some character $\chi_m(z) = z^m$ with $m \geq 0$.

If $\rho|_{\mathbb{T}_c} : \mathbb{T}_c \to U(H)$ acts via the character $\chi_k(z) = z^k$, then ρ is a representation of level k.

- $R_k(LG)$ - formal differences of isomorphism classes of positive energy representations of LG at a fixed level $k \in \mathbb{Z}$.

Properties

- $R_k(LG)$ is a ring with respect to the fusion product.
- It is the quotient ring of $R(G)$ by the fusion ideal.
- The category $Rep_k(LG)$ of positive energy representations of level k is a modular tensor category.

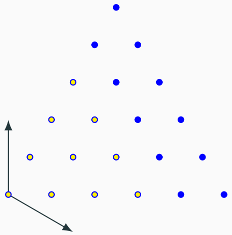

Figure 3: $R(SU(3))$ vs. $R_3(LSU(3))$.

Loop Groups and the Verlinde Ring

G. Segal:
In fact, it is not much of an exaggeration to say that the mathematics of two-dimensional quantum field theory is almost the same thing as the representation theory of loop groups.

- Each modular tensor category gives rise to a 3*d*-TQFT (Reshetikhin-Turaev).
- $Rep_k(LG)$ gives Chern-Simons theory.
- The vector space that this theory associates to $S^1 \times S^1$ is a (complexified) K-theory group.
- The corresponding K-theory group is part of a 2*d*-TQFT.

The Verlinde Ring and Twisted K-theory

Theorem (Freed, Hopkins, Teleman)

Let G be a simply connected compact Lie group and let LG be the free loop group on G. Then we have

$$ {}^{\tau(k)}K_G^{\dim(G)}(G^{adj}) \cong R_k(LG) \ . $$

K-theory and homotopy theory

- $K^1(X) = [X, U_\infty]$,
- $K^0(X) = [X, BU_\infty \times \mathbb{Z}]$,

If we define $KU^\Omega_{2k} = BU_\infty \times \mathbb{Z}$ and $KU^\Omega_{2k+1} = U_\infty$, then there are homotopy equivalences

$$ KU^\Omega_n \to \Omega KU^\Omega_{n+1} \ . $$

This is an Ω-spectrum representing K-theory. K-theory can also be represented by a ring spectrum KU_*!

Twisted K-theory by Analogy

Idea of twisted K-theory:

- R commutative ring with unit
- X topological space
- $\mathcal{R} \to X$ locally trivial bundle of free R-modules of rank 1
- $C(X, \mathcal{R})$ is a module over $C(X, R)$ and "loc. indistinguishable" from it

$$\text{Bundles } \mathcal{R}/\text{iso} \quad \stackrel{1:1}{\longleftrightarrow} \quad [X, BGL_1(R)]$$

Now replace ring R by ring spectrum KU_*!

- Twists are classified by $\tau \colon X \to BGL_1(KU)$

$$\begin{array}{ccc} \tau^* E_* & \longrightarrow & E_* \\ \downarrow & & \downarrow \\ X & \xrightarrow[\tau]{} & BGL_1(KU) \end{array}$$

Twisted K-theory (contd.)

Given $\tau \colon X \to BGL_1(KU)$ we define

$$^{\tau}K^n(X) = [X, \tau^* E_n] \ ,$$

where we now take homotopy classes of sections of $\tau^* E \to X$.

Caveats:

- $GL_1(KU)$ is not a group a priori, but an infinite loop space!
- E_* needs to be constructed carefully!
  (May-Sigurdsson, Ando-Blumberg-Gepner,
  Ando-Blumberg-Gepner-Hopkins-Rezk, ...)

Twisted K-theory via Operator Algebras

What does $BGL_1(KU)$ look like?

$$BGL_1(KU) \simeq B(BU \times \{\pm 1\})_{\otimes}$$

where $BU \times \{\pm 1\}$ is the classifying space of virtual line bundle of dimension ± 1.

- Have a natural map $BU(1) \to BU \times \{\pm 1\}$ giving rise to

$$BBU(1) \to BGL_1(KU)$$

- $BBU(1) \simeq BPU(H) \simeq B\mathrm{Aut}(\mathbb{K})$.

If τ factors through $BBU(1)$, then it classifies a bundle $\mathcal{K}_\tau \to X$ of compact operators and

$$^{\tau}K^n(X) \cong K_n(C(X, \mathcal{K}_\tau)) \ .$$

Twisted K-theory via Operator Algebras

Have $\mathcal{B}un_{\mathbb{K}}(X) \cong H^3(X, \mathbb{Z}) \cong [X, BBU(1)]$ (Dixmier-Douady).

Is there a C^*-algebra A, such that

$$\mathcal{B}un_A(X) \cong [X, BGL_1(KU)] \text{ ?}$$

Let D be a unital C^*-algebra, let X be a compact metrizable space

D self-absorbing $\rightsquigarrow$ $\mathcal{B}un_{D\otimes\mathbb{K}}(X)$ is a semigroup w.r.t. $\otimes$

D strongly self-absorbing $\rightsquigarrow$ $\mathcal{B}un_{D\otimes\mathbb{K}}(X)$ is a monoid w.r.t. $\otimes$

Theorem (Dadarlat, P.)

If D is strongly self-absorbing, then $\mathcal{B}un_{D\otimes\mathbb{K}}(X)$ is a group. In particular,

$$\mathcal{B}un_{\mathcal{O}_\infty\otimes\mathbb{K}}(X) \cong [X, BGL_1(KU)]$$
$$\mathcal{B}un_{M_n^{\otimes\infty}\otimes\mathbb{K}}(X) \cong [X, BBU_{\otimes}[\tfrac{1}{n}]]$$

Let's do the twist ...

Theorem (Freed, Hopkins, Teleman)

$$ {}^{\tau(k)}K_G^{\dim(G)}(G^{adj}) \cong R_k(LG) \ . $$

- In this theorem $\tau(k)$ factors through $BBU(1)$. What is the associated bundle of compact operators $\mathcal{K}_\tau \to G$?
- Is there a generalisation of τ that "sees" all of $BGL_1(KU)$, but preserves as much of the structure of $R_k(LG)$ as possible?
- G-equivariance?

Exponential functors

- $\mathcal{V}ect_{\mathbb{C}}^{fin}$ - fin. dim. complex inner product spaces and linear maps
- $\mathcal{V}ect_{\mathbb{C}}^{iso}$ - same objects but with unitary isomorphisms

Definition

An exponential functor on $\mathcal{V}ect_{\mathbb{C}}^{fin}$ consists of a triple (F, κ, ι), where

- $F \colon \mathcal{V}ect_{\mathbb{C}}^{fin} \to \mathcal{V}ect_{\mathbb{C}}^{fin}$ is a unitary functor,
- $\kappa_{V,W} \colon F(V \oplus W) \to F(V) \otimes F(W)$ is a natural isomorphism,
- $\iota \colon F(0) \to \mathbb{C}$ is another natural isomorphism,

such that the obvious associativity and unitality diagrams commute.

Exponential functors

Example

- $F = \left(\bigwedge^*\right)^{\otimes m}$ for any $m \in \mathbb{N}_0$,
- $F = \left(\bigwedge^{top}\right)^{\otimes m}$ for any $m \in \mathbb{N}_0$ on $\mathcal{V}ect_{\mathbb{C}}^{iso}$,
- Fix $W \in obj(\mathcal{V}ect_{\mathbb{C}}^{fin})$, then

$$F^W(V) = \bigoplus_{k=0}^{\infty} W^{\otimes k} \otimes \bigwedge^k(V) \, .$$

- non-example: symmetric algebra
- classification of polynomial exponential functors via involutive R-matrices (based on Lechner-P-Wood)

A Higher Twist over $SU(n)$ (jt. with D. Evans)

Input: Exponential functor F on $\mathcal{V}ect_{\mathbb{C}}^{iso}$.

Output:

- groupoid $\mathcal{G}$ with a map $\mathcal{G} \to G$, where $G = SU(n)$,
- saturated Fell bundle $\mathcal{E} \to \mathcal{G}$, i.e. a bundle of D-D-Morita equivalences plus an associative multiplication

$$\mathcal{E}_{g_1} \otimes_D \mathcal{E}_{g_2} \to \mathcal{E}_{g_1 \cdot g_2}$$

- ...with $\mathcal{E}|_{\mathcal{G}^{(0)}} = \mathcal{G}^{(0)} \times D$, where

$$D = \mathrm{End}(F(\mathbb{C}^n))^{\otimes \infty} \ .$$

Theorem (Evans-Pennig)

$$C^*(\mathcal{E}) \otimes \mathbb{K} \cong C(G, \mathcal{A}) \ ,$$

where $\mathcal{A} \to G$ is a bundle with fibre $D \otimes \mathbb{K}$. We have G-actions on $\mathcal{G}$, $\mathcal{E}$ and $C^*(\mathcal{E})$ compatible with conjugation action of G on G.

A Higher Twist over $SU(n)$ (jt. with D. Evans)

Remarks:

- For $F = \left(\bigwedge^{top}\right)^{\otimes m}$ we have $D = \mathbb{C}$ and construction boils down to the m-th tensor power of the basic gerbe a la Murray-Stevenson, which represents classical twist.
- Can compute ${}^{\mathcal{E}}K^n_G(G) := K^G_n(C^*(\mathcal{E}))$ ...
  - for $n = 2$ and all F,
  - for $n = 3$ and all F after rationalisation.

  In all these examples ${}^{\mathcal{E}}K^{\dim(G)}_G(G)$ is still a ring!
- Even better: Sometimes seem to get fusion rings from loop group CFTs, e.g. for $n = 2$ and $F = \left(\bigwedge^*\right)^{\otimes(2m+1)}$.

Open Questions

- Does this construction non-equivariantly correspond to

$$SU(n) \to SU_\infty \simeq BBU_\oplus \to BBU_\otimes[\tfrac{1}{k}] \quad ?$$

- Is there a ring structure in general? Intrinsic reason for this?
- Is there such a nice description of the generalised rational Dixmier-Douady classes?
- What is the correct replacement of the right hand side in the FHT theorem?

Thank you!