Section A

(Answer ALL Questions)

1. Consider the vector space $P_2(\mathbb{R})$ over $\mathbb{R}$. Let $\beta = \{1, x, x^2\}$ be the basis of $P_2(\mathbb{R})$ given by monomials. Let $T\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation whose action on the basis β is as follows:

$$\begin{aligned} T(1) &= 1 + x \\ T(x) &= 1 - x \\ T(x^2) &= 3x^2 \end{aligned}$$

   (a) Determine the matrix representation $A = [T]_\beta^\beta$ of T. [3]

   (b) For the matrix A determined in (a) compute $\operatorname{adj}(A)$, A^{-1} and $\det(A)$. [10]

   (c) Is T an isomorphism? Justify your answer. [2]

2. Let V, W_1 and W_2 be finite-dimensional vector spaces over $\mathbb{R}$ and let $\mathcal{L}(V, W_i)$ be the vector space of linear transformations $T\colon V \to W_i$. Let $S\colon W_1 \to W_2$ be an isomorphism. Consider the map

$$\varphi_S\colon \mathcal{L}(V, W_1) \to \mathcal{L}(V, W_2) \quad , \quad T \mapsto S \circ T \ .$$

   (a) Show that φ_S is a linear transformation. [6]

   (b) Prove that φ_S is an isomorphism. [6]

   (c) Express the rank of φ_S in terms of $\dim(V)$ and $\dim(W_i)$. [3]

3. (a) Let $T\colon V \to V$ be a linear transformation on a vector space over the field F. Define what it means for $v \in V$ to be an eigenvector of T. [5]

   (b) Let $T\colon V \to V$ be a nilpotent linear transformation. Suppose that $\lambda \in F$ is an eigenvalue of T. Show that $\lambda = 0$. [5]

4. (a) For a matrix $A \in M_{n\times n}(\mathbb{C})$ with entries $a_{ij} \in \mathbb{C}$ define A^* to be the matrix with $(A^*)_{ij} = \overline{a_{ji}}$, i.e. $A^* = \overline{A^t}$. Show that [5]

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)} \ .$$

   (b) Let $A \in M_{n\times n}(\mathbb{C})$ be a matrix with $A^* = A$. Use part (a) to prove that in this case the trace $\operatorname{tr}(A)$ is a real number. [5]

5. Consider the following system of linear equations over $\mathbb{R}$:

$$\begin{aligned} x_1 + 3x_2 + 5x_3 + 9x_4 &= 0 \\ 2x_1 + 4x_2 + 6x_3 + 7x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= 0 \end{aligned}$$

   (a) Write down the augmented matrix $(A \mid b)$ of this system of linear equations. [3]

   (b) Bring the augmented matrix into reduced row echelon form. [5]

   (c) Write down the solution set of this system of linear equations. [4]

   (d) Determine the rank of the matrix A in the system of linear equations. [3]

6. Let V be a finite-dimensional vector space over $\mathbb{R}$ and let $V^* = \mathcal{L}(V, \mathbb{R})$ be the dual space of V. For a linear subspace $U \subset V$ define

$$U^\dagger = \{\varphi \in V^* \mid \varphi(u) = 0 \text{ for all } u \in U\} \subset V^*$$

   Let $U_1, U_2 \subset V$ be linear subspaces.

   (a) Show that $(U_1 + U_2)^\dagger = U_1^\dagger \cap U_2^\dagger$. [6]

   (b) Prove that $U_1^\dagger + U_2^\dagger \subseteq (U_1 \cap U_2)^\dagger$. [4]

Section B

(Answer THREE Questions)

7. Let $A \in M_{2\times 2}(\mathbb{C})$ be the following matrix

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$$

   (a) Find the eigenvalues of A and bases for the corresponding eigenspaces. [13]

   (b) Determine all generalised eigenspaces for $L_A \colon \mathbb{C}^2 \to \mathbb{C}^2$. [4]

   (c) Is this matrix diagonalisable? Write down a Jordan normal form for A. [8]

8. Let $A \in M_{3\times 3}(\mathbb{C})$ be the matrix with complex entries given below:

$$A = \begin{pmatrix} i & i & -i \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

   (a) Compute $\det(A)$, $\det(A^2)$ and $\det(3A)$ and write down A^t and $A^* = \overline{A^t}$. [5]

   (b) Let $B \in M_{3\times 3}(\mathbb{C})$ be an invertible matrix. Use properties of the determinant to show that $\det(B^{-1}) = \det(B)^{-1}$ and [10]

$$\det(BAB^{-1}) = \det(A) \ .$$

   (c) Use part (a) and part (b) to prove that there is no invertible matrix $B \in M_{3\times 3}(\mathbb{C})$ with the property that BAB^{-1} only has real entries. [10]

9. Let V be a finite-dimensional vector space over $\mathbb{R}$ equipped with an inner product

$$\langle\,\cdot\,,\,\cdot\,\rangle \colon V \times V \to \mathbb{R}$$

   and denote by $\|v\| = \sqrt{\langle v, v\rangle}$ the norm of $v \in V$.

   (a) Define what it means for a subset β of V to be an orthonormal basis for V. [5]

   (b) Show that if $u, v \in V$ have the same norm, then $u + v$ and $u - v$ are orthogonal to each other. [5]

   (c) Let $u, v \in V$ and suppose that $\|u\| = \|v\| = 1$ and $\langle u, v\rangle = 1$. Show that in this case we have $u = v$. [7]

   (d) Describe what happens if the Gram–Schmidt orthonormalisation process is applied to a set $\{v_1, \ldots, v_n\} \subset V$ that is linearly dependent. [8]

10. Let V be a finite-dimensional vector space over the field F and let $T\colon V \to V$ be a linear transformation with the property $T^2 = T \circ T = T$.

    (a) Prove that the only possible eigenvalues of T are 0 and 1. [4]

    (b) Show that $\ker(T)$ and $\operatorname{Im}(T)$ are the only possible eigenspaces of T. [9]

    (c) Prove that $V = \ker(T) \oplus \operatorname{Im}(T)$. [7]

    (d) Use (b) and (c) to show that T is diagonalisable. [5]