Mock exam for the lecture
MA3008 – Algebraic Topology

Mock exam Spring Semester 2017

Exercise 1 (Metric spaces, Continuity).

a) Give the definition of a metric space.

b) Define what it means for a map $f\colon X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) to be continuous at $x_0 \in X$.

c) Let (X, d_X) be a metric space and let $\mathbb{R}$ be equipped with the usual metric. Let $a \in X$ and consider the map $f_a\colon X \to \mathbb{R}$ given by $f_a(x) = d_X(x, a)$. Show that f_a is continuous.

d) Let (X, d_X) be a metric space. Let $d'(x, y) = \ln(1 + d_X(x, y))$ for all $x, y \in X$. Show that this defines a new metric on X, i.e. that (X, d') is also a metric space. You may use without proof that ln is monotonically increasing.

Exercise 2 (Basis of a topology).

a) Give the definition of a basis $\mathcal{B}$ of a topology $\mathcal{T}$.

b) Let X be a set. State the properties that need to hold in order for a family $\mathcal{B}$ of subsets of X to be a basis of a topology $\mathcal{T}_{\mathcal{B}}$.

c) Let $X = \mathbb{R}$ and consider the family $\mathcal{B}$ of all one-point subsets, i.e.

$$\mathcal{B} = \{\{x\} \mid x \in \mathbb{R}\} \ .$$

Show that this is the basis of a topology on X.

d) Let $X = \mathbb{R}$ and let $\mathcal{B}$ be as in c). Let X be equipped with the topology $\mathcal{T}_{\mathcal{B}}$ obtained from the basis $\mathcal{B}$. Let $Y = \mathbb{R}$ be equipped with the metric topology $\mathcal{T}(d)$ with respect to the metric $d(x, y) = |x - y|$. Prove that the map $f\colon X \to Y$ given by $f(x) = x$ is continuous.

e) Let $(X, \mathcal{T}_{\mathcal{B}})$, $(Y, \mathcal{T}(d))$ be the topological spaces from d) and let $f\colon X \to Y$ be the continuous map from d). Show that it is bijective, but not a homeomorphism.

Exercise 3 (Subspace topology, Convergence of sequences).

a) Let $(X, \mathcal{T}_X)$ be a topological space and let $Y \subset X$ be a subset. Show that the family of subsets given by

$$\mathcal{T}_{Y \subset X} = \{U \subset Y \mid U = V \cap Y \text{ for some } V \in \mathcal{T}_X\}$$

is a topology on Y. (This is the subspace topology.)

b) Let $X = \mathbb{R}$ be equipped with the metric topology from the metric $d(x, y) = |x - y|$. Let

$$Y = \left\{x \in \mathbb{R} \mid x = \frac{1}{n}, n \in \mathbb{N}\right\} \cup \{0\} \subset \mathbb{R}$$

be equipped with the subspace topology. Sketch a picture of this topological space.

c) Let X and Y be the topological spaces from b). Let $U \subset Y$ be an open subset with $0 \in U$. Show that there is $x \in U$ with $x \neq 0$. Deduce that the subspace topology $\mathcal{T}_{Y \subset X}$ on Y is not the same as the discrete topology $\mathcal{T}_{\text{dis}}$ on Y.

d) Let $(Z, \mathcal{T}_Z)$ be another topological space, let Y be as in c) and let $f \colon Y \to Z$ be a continuous map. Define $a_n = f(\frac{1}{n})$ for $n \in \mathbb{N}$. Show that the sequence $(a_n)_{n \in \mathbb{N}}$ converges to $a = f(0)$ in Z.

Exercise 4 (Quotient topology).

a) Let $(X, \mathcal{T})$ be a topological space and let $\sim$ be an equivalence relation on X. Give the definition of the quotient topology on $X/\sim$.

b) Let $X = \mathbb{R}$ be equipped with the metric topology $\mathcal{T}(d)$ with respect to $d(x, y) = |x - y|$. Let $\sim$ be the relation defined by

$$x_1 \sim x_2 \qquad \Leftrightarrow \qquad x_1 - x_2 \in \mathbb{Z} \ .$$

Show that this is an equivalence relation.

c) Let $[0, 1) \subset \mathbb{R}$ be equipped with the subspace topology and let X and $\sim$ be as in b). Prove that the map $f \colon [0, 1) \to X/\sim$ given by $f(x) = [x]$ is continuous and bijective, but not a homeomorphism.