Mock exam for the lecture
MA3008 – Algebraic Topology

Second Mock Exam Spring Semester 2017

Exercise 1 (Metric spaces).
Let (X, d) be a metric space.

a) Give the definition of the metric topology $\mathcal{T}(d)$.

b) Let $x \in X$ and let $r > 0$. Show that the subset $B_r(x) = \{x' \in X \mid d(x', x) < r\}$ is open with respect to $\mathcal{T}(d)$.

c) Let $d'(x, y) = \sqrt{d(x, y)}$ for all $x, y \in X$. Show that this defines a new metric on X, i.e. that (X, d') is also a metric space.

d) Let $d' \colon X \times X \to \mathbb{R}$ be the metric from part (c). Prove that $\mathcal{T}(d') = \mathcal{T}(d)$.

Exercise 2 (Connectedness and continuous maps).
Let $\mathbb{R}$ and $\mathbb{R}^2$ be equipped with their standard metric topologies and let $S^1 \subset \mathbb{R}^2$ and $S^0 = \{-1, 1\} \subset \mathbb{R}$ be equipped with their subspace topologies.

a) Give the definitions of path-connected and of connected.

b) Show that the subspace topology $S^0 \subset \mathbb{R}$ is the same as the discrete topology on S^0.

c) Let $f \colon S^1 \to S^0$ be a continuous map. Prove that it has to be equal either to the constant map with value 1 or the constant map with value -1.

d) Suppose that $f \colon S^1 \to \mathbb{R}$ is a continuous map. Prove that there has to be a point $x \in S^1$, such that $f(x) = f(-x)$.

   Hint: Suppose that there is no such point and consider the map

$$g \colon S^1 \to S^0 \quad ; \quad g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|} \ .$$

Exercise 3 (Compact spaces).

a) Give the definition of compact topological space and the definition of Hausdorff space.

b) Let $(X, \mathcal{T}_X)$ be a compact topological space and let $A \subset X$ be a closed subspace. Prove that A is compact.

c) Let $\mathbb{R}$ be equipped with its standard metric topology. Show that if X is compact and $f\colon X \to \mathbb{R}$ is a continuous map, then f is bounded and takes on a minimum and a maximum value.

d) Let $(X, \mathcal{T}_X)$ be a topological space, let $A \subset X$ and $B \subset X$ be compact subspaces. Show that $A \cup B$ is a compact subspace as well.

Exercise 4 (Homotopy equivalence).

a) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces and let $f_1, f_2\colon X \to Y$ be two continuous maps. Define what is meant by: f_1 is homotopic to f_2.

b) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. Define what it means for a continuous map $f\colon X \to Y$ to be a homotopy equivalence.

c) Let $Y = \mathbb{R}^3 \setminus \{(0,0,z) \in \mathbb{R}^3 \mid z \in \mathbb{R}\} \subset \mathbb{R}^3$ be equipped with the subspace topology, i.e. Y is the complement of the z-axis. Show that the map $f\colon Y \to \mathbb{R}^2 \setminus \{0\}$ given by $f(x,y,z) = (x,y)$ is a homotopy equivalence.

d) Let $Y \subset \mathbb{R}^3$ be the subspace from part (b) and let $x_0 = (1,0,0) \in \mathbb{R}^3$. Prove that $\pi_1(Y, x_0) \cong \mathbb{Z}$. You may use without proof that $\pi_1(S^1, z_0) \cong \mathbb{Z}$ for any basepoint $z_0 \in S^1$ as shown in the lecture.