Exercises for the lecture
MA3008 – Algebraic Topology

Solutions of Mock Exam 01 Spring Semester 2017

Exercise 1 (Metric spaces, Continuity).

a) Give the definition of a metric space.

b) Define what it means for a map $f\colon X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) to be continuous at $x_0 \in X$.

c) Let (X, d_X) be a metric space and let $\mathbb{R}$ be equipped with the usual metric. Let $a \in X$ and consider the map $f_a\colon X \to \mathbb{R}$ given by $f_a(x) = d_X(x, a)$. Show that f_a is continuous.

d) Let (X, d_X) be a metric space. Let $d'(x, y) = \ln(1 + d_X(x, y))$ for all $x, y \in X$. Show that this defines a new metric on X, i.e. that (X, d') is also a metric space. You may use without proof that ln is monotonically increasing.

Solution: A metric space (X, d) consists of a set X together with a function

$$d\colon X \times X \to \mathbb{R} ,$$

which satisfies the following properties:

a) for all $x, y \in X$ we have $d(x, y) \geq 0$,

b) $d(x, y) = 0 \quad \Leftrightarrow \quad x = y$,

c) for all $x, y \in X$ we have $d(x, y) = d(y, x)$,

d) the triangle inequality holds: $d(x, z) \leq d(x, y) + d(y, z)$.

A function $f\colon X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is continuous at the point $x_0 \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$, such that $d_X(x, x_0) < \delta$ implies $d_Y(f(x), f(x_0)) < \epsilon$.
Let $a \in X$ and consider $f_a(x) = d_X(x, a)$ as in part c). Fix $y \in X$. With $Y = \mathbb{R}$ and $d_Y(s, t) = d_{\mathbb{R}}(s, t) = |s - t|$ we have

$$d_{\mathbb{R}}(f_a(x), f_a(y)) = |f_a(x) - f_a(y)| = |d_X(x, a) - d_X(y, a)| .$$

So we need to find an estimate of the right hand side in terms of $d_X(x,y)$. By the triangle inequality we have

$$d_X(x,a) \leq d_X(x,y) + d_X(y,a) \quad \Leftrightarrow \quad d_X(x,a) - d_X(y,a) \leq d_X(x,y) \ .$$

Switching the roles of x and y we also get: $d_X(y,a) - d_X(x,a) \leq d_X(x,y)$. Hence,

$$|d_X(x,a) - d_X(y,a)| \leq d_X(x,y) \ .$$

Given $\epsilon > 0$ we can now choose $\delta = \epsilon$ and obtain that $d_X(x,y) < \delta = \epsilon$ implies

$$d_{\mathbb{R}}(f_a(x), f_a(y)) = |d_X(x,a) - d_X(y,a)| \leq d_X(x,y) < \epsilon \ .$$

Therefore f_a is continuous at $y \in X$.
For part d) we need to check that d' satisfies the properties of a metric: First observe that $1 + d_X(x,y) \geq 1$, $\ln(1) = 0$ and ln is monotonically increasing. Therefore we have $d'(x,y) \geq 0$. Suppose that $x, y \in X$ are two points with $d'(x,y) = 0$, then

$$\ln(1 + d_X(x,y)) = 0 \quad \Leftrightarrow \quad 1 + d_X(x,y) = 1 \quad \Leftrightarrow \quad d_X(x,y) = 0 \quad \Leftrightarrow \quad x = y \ .$$

Symmetry is a direct consequence of the symmetry of $d_X(x,y)$:

$$d'(x,y) = \ln(1 + d_X(x,y)) = \ln(1 + d_X(y,x)) = d'(y,x) \ .$$

To check the triangle inequality, let $x, y, z \in X$. Then we have

$$\begin{aligned} & d'(x,y) + d'(y,z) \\ = \ & \ln(1 + d_X(x,y)) + \ln(1 + d_X(y,z)) \\ = \ & \ln((1 + d_X(x,y))(1 + d_X(y,z))) \\ = \ & \ln(1 + d_X(x,y) + d_X(y,z) + d_X(x,y)d_X(y,z)) \\ \geq \ & \ln(1 + d_X(x,y) + d_X(y,z)) \\ \geq \ & \ln(1 + d_X(x,z)) = d'(x,z) \end{aligned}$$

where we have used that $d_X(x,y)d_X(y,z) \geq 0$ in the second to last row and the fact that ln is monotonically increasing together with the triangle inequality for d_X in the last one.

Exercise 2 (Basis of a topology).

a) Give the definition of a basis $\mathcal{B}$ of a topology $\mathcal{T}$.

b) Let X be a set. State the properties that need to hold in order for a family $\mathcal{B}$ of subsets of X to be a basis of a topology $\mathcal{T}_{\mathcal{B}}$.

c) Let $X = \mathbb{R}$ and consider the family $\mathcal{B}$ of all one-point subsets, i.e.

$$\mathcal{B} = \{\{x\} \mid x \in \mathbb{R}\} \ .$$

Show that this is the basis of a topology on X.

d) Let $X = \mathbb{R}$ and let $\mathcal{B}$ be as in c). Let X be equipped with the topology $\mathcal{T}_{\mathcal{B}}$ obtained from the basis $\mathcal{B}$. Let $Y = \mathbb{R}$ be equipped with the metric topology $\mathcal{T}(d)$ with respect to the metric $d(x, y) = |x - y|$. Prove that the map $f\colon X \to Y$ given by $f(x) = x$ is continuous.

e) Let $(X, \mathcal{T}_{\mathcal{B}})$, $(Y, \mathcal{T}(d))$ be the topological spaces from d) and let $f\colon X \to Y$ be the continuous map from d). Show that it is bijective, but not a homeomorphism.

Solution A family $\mathcal{B}$ of subsets of a topological space $(X, \mathcal{T})$ is a basis of the topology $\mathcal{T}$ if every set $U \in \mathcal{T}$ is a union of sets from $\mathcal{B}$.
Let X be a set and let $\mathcal{B}$ be a family of subsets of X. Let

$$\mathcal{T}_{\mathcal{B}} = \{U \subset X \mid \text{for all } x \in U \text{ there exists } B \in \mathcal{B} \text{ such that } x \in B \subset U\} \ .$$

Then $\mathcal{T}_{\mathcal{B}}$ is a topology with basis $\mathcal{B}$ if and only if $\mathcal{B}$ has the following two properties:

a) For each $x \in X$ there exists a $B \in \mathcal{B}$ with $x \in B$.

b) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then there exists a $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$.

Now let $\mathcal{B} = \{\{x\} \mid x \in \mathbb{R}\}$ as in part c) of the Exercise. To see that this family is the basis of a topology $\mathcal{T}_{\mathcal{B}}$ on X, we need to check that it has the properties a) and b) above. For $x \in X$ we can take $B = \{x\}$ and we have $x \in \{x\} = B$, which is property a). Now let $B_1, B_2 \in \mathcal{B}$, such that there is $x \in B_1 \cap B_2$. Using the definition of $\mathcal{B}$ there must be $x_1, x_2 \in X$, such that $B_1 = \{x_1\}$ and $B_2 = \{x_2\}$. Therefore $B_1 \cap B_2$ is either empty or $x_1 = x_2$, in which case $B_1 \cap B_2 = \{x_1\} = \{x_2\}$. Since we know that $x \in B_1 \cap B_2$, it can not be empty and we must have $x = x_1 = x_2$. Choose $B_3 = \{x\}$. Then we have

$$x \in B_3 \subset B_1 \cap B_2 \ ,$$

which is property b). Hence, $\mathcal{B}$ is in fact the basis of a topology $\mathcal{T}_{\mathcal{B}}$ defined as above. Some further thought shows that $\mathcal{T}_{\mathcal{B}}$ is the discrete topology on $\mathbb{R}$.

Let $X = \mathbb{R}$ be equipped with the topology $\mathcal{T}_{\mathcal{B}}$. Let $Y = \mathbb{R}$ be equipped with the topology $\mathcal{T}(d)$ from the metric $d(x, y) = |x - y|$ and consider the map $f\colon X \to Y$ with $f(x) = x$. On the underlying sets this is just the identity map. However, we have chosen different topologies on X and on Y. Therefore it is not obvious that f is continuous. To see that it is, it suffices to check that $f^{-1}(B_r(x_0)) \subset X$ is open for every $r > 0$ and $x_0 \in X$, since the sets $B_r(x_0)$ form a basis for the metric topology $\mathcal{T}(d)$. Note that $f^{-1}(B_r(x_0)) = B_r(x_0)$. According to the definition of $\mathcal{T}_{\mathcal{B}}$ we have to check that we can find for every $x \in B_r(x_0)$ a basis element $B \in \mathcal{B}$, such that $x \in B \subset B_r(x_0)$. To achieve this we can choose $B = \{x\}$.

To see that $f\colon X \to Y$ is bijective, we can write down the inverse map, which is given by $g\colon Y \to X$ with $g(x) = x$. On the underlying sets it is (and has to be!) the identity map again. However, g is not continuous: In fact, $g^{-1}(\{x\}) = \{x\}$ and $\{x\} \in \mathcal{B} \subset \mathcal{T}_{\mathcal{B}}$ is open in $(X, \mathcal{T}_{\mathcal{B}})$, whereas it is not open in $(Y, \mathcal{T}(d))$, since every non-empty open subset in Y has to contain an open interval (a, b) for some $a, b \in \mathbb{R}$ with $a < b$.

Exercise 3 (Subspace topology, Convergence of sequences).

a) Let $(X, \mathcal{T}_X)$ be a topological space and let $Y \subset X$ be a subset. Show that the family of subsets given by

$$\mathcal{T}_{Y\subset X} = \{U \subset Y \mid U = V \cap Y \text{ for some } V \in \mathcal{T}_X\}$$

is a topology on Y. (This is the subspace topology.)

b) Let $X = \mathbb{R}$ be equipped with the metric topology from the metric $d(x, y) = |x - y|$. Let

$$Y = \left\{x \in \mathbb{R} \mid x = \frac{1}{n}, n \in \mathbb{N}\right\} \cup \{0\} \subset \mathbb{R}$$

be equipped with the subspace topology. Sketch a picture of this topological space.

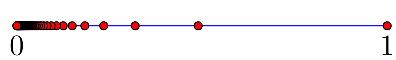
c) Let X and Y be the topological spaces from b). Let $U \subset Y$ be an open subset with $0 \in U$. Show that there is $x \in U$ with $x \neq 0$. Deduce that the subspace topology $\mathcal{T}_{Y\subset X}$ on Y is not the same as the discrete topology $\mathcal{T}_{\text{dis}}$ on Y.

d) Let $(Z, \mathcal{T}_Z)$ be another topological space, let Y be as in c) and let $f: Y \to Z$ be a continuous map. Define $a_n = f(\frac{1}{n})$ for $n \in \mathbb{N}$. Show that the sequence $(a_n)_{n\in\mathbb{N}}$ converges to $a = f(0)$ in Z.

Solution We have $Y = X \cap Y$ and $X \in \mathcal{T}_X$, which implies $Y \in \mathcal{T}_{Y\subset X}$. Likewise, $\emptyset = \emptyset \cap Y$ and therefore $\emptyset \in \mathcal{T}_{Y\subset X}$.
Suppose that $U_1, U_2 \in \mathcal{T}_{Y\subset X}$. This means that there are sets $V_1, V_2 \in \mathcal{T}_X$, such that $U_i = V_i \cap Y$ for $i \in \{1, 2\}$. Since $\mathcal{T}_X$ is a topology on X, we have $V_1 \cap V_2 \in \mathcal{T}_X$ and $U_1 \cap U_2 = (V_1 \cap Y) \cap (V_2 \cap Y) = (V_1 \cap V_2) \cap Y$.
Let I be a set and let $U_i \in \mathcal{T}_{Y\subset X}$ for $i \in I$. By the definition of $\mathcal{T}_{Y\subset X}$ this means that for every $i \in I$ there is $V_i \in \mathcal{T}_X$, such that $U_i = V_i \cap Y$. Since $\mathcal{T}_X$ is a topology on X, we have $\bigcup_{i\in I} V_i \in \mathcal{T}_X$. Now note that

$$\bigcup_{i\in I} U_i = \bigcup_{i\in I}(V_i \cap Y) = \left(\bigcup_{i\in I} V_i\right) \cap Y \ .$$

This proves that $\mathcal{T}_{Y\subset X}$ is a topology on Y.
A sketch of the topological space Y from b) is shown below:



Let $U \subset Y$ be an open subset with $0 \in U$ as in part c) of the exercise. Since U is open in the subspace topology, there is an open subset $V \subset \mathbb{R}$ with $U = V \cap Y$. Since $0 \in V$ and V is open with respect to the metric d, we can find an $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subset V$. Since the sequence $a_n = \frac{1}{n}$ converges to 0 for $n \to \infty$, there is $N \in \mathbb{N}$, such that the open set V will contain all elements a_m for $m > N$. In particular, there will be at least one $a_m \in V$. Since $\frac{1}{m} \in Y$, we have $\frac{1}{m} \in V \cap Y = U$. But $\frac{1}{m} \neq 0$, hence we have solved the first part of c). Our observations show that the subset $\{0\} \subset Y$ is not open. However, in the discrete topology $\mathcal{T}_{\text{dis}}$ all subsets of Y are open. Therefore $\mathcal{T}_{Y \subset X} \neq \mathcal{T}_{\text{dis}}$.

Let $f \colon Y \to Z$ be a continuous map as in d). Let $a_n = f(\frac{1}{n}) \in Z$ and $a = f(0) \in Z$. Let $U' \subset Z$ be a neighbourhood of a. This means that it contains an open subset $U \subset Z$ with $U \subset U'$ and $a \in U$. Since f is continuous, we obtain that $f^{-1}(U)$ is open in Y. Note that $0 \in f^{-1}(U)$. Just as in the last paragraph we have that there is an $N \in \mathbb{N}$, such that $\frac{1}{m} \in f^{-1}(U)$ for all $m > N$. But this means that $a_m = f(\frac{1}{m}) \in U \subset U'$ for all $m > N$. Hence, $(a_n)_{n \in \mathbb{N}}$ converges to a in Z.

Exercise 4 (Quotient topology).

a) Let $(X, \mathcal{T})$ be a topological space and let $\sim$ be an equivalence relation on X. Give the definition of the quotient topology on $X/\sim$.

b) Let $X = \mathbb{R}$ be equipped with the metric topology $\mathcal{T}(d)$ with respect to $d(x, y) = |x - y|$. Let $\sim$ be the relation defined by

$$x_1 \sim x_2 \qquad \Leftrightarrow \qquad x_1 - x_2 \in \mathbb{Z} \ .$$

Show that this is an equivalence relation.

c) Let $[0, 1) \subset \mathbb{R}$ be equipped with the subspace topology and let X and $\sim$ be as in b). Prove that the map $f : [0, 1) \to X/\sim$ given by $f(x) = [x]$ is continuous and bijective, but not a homeomorphism.

Solution Let $(X, \mathcal{T}_X)$ be a topological space and let $\sim$ be an equivalence relation on X. Let $q : X \to X/\sim$ be given by $q(x) = [x]$. The quotient topology $\mathcal{T}_{X/\sim}$ on $X/\sim$ is defined as follows:

$$\mathcal{T}_{X/\sim} = \left\{ U \subset X/\sim \ \mid q^{-1}(U) \in \mathcal{T}_X \right\} \ .$$

To check that the relation $\sim$ from part b) is an equivalence relation note the following:

- $x - x = 0 \in \mathbb{Z}$ implies $x \sim x$, which is the reflexivity of $\sim$.
- If we have $x_1 \sim x_2$, then $x_1 - x_2 \in \mathbb{Z}$. But this means that $x_2 - x_1 = -(x_1 - x_2) \in \mathbb{Z}$ and therefore $x_2 \sim x_1$. This is the symmetry of $\sim$.
- If $x_1 \sim x_2$ and $x_2 \sim x_3$, then we have $x_1 - x_3 = (x_1 - x_2) + (x_2 - x_3) \in \mathbb{Z}$, which implies $x_1 \sim x_3$. Therefore $\sim$ is also transitive.

Let $f : [0, 1) \to X/\sim = \mathbb{R}/\sim$ be the map from c). Since $[0, 1)$ carries the subspace topology, the inclusion map $\hat{f} : [0, 1) \to \mathbb{R}$ is continuous. Let $q : \mathbb{R} \to \mathbb{R}/\sim$ be the quotient map and note that $f = q \circ \hat{f}$. Since f is the composition of two continuous maps, it is continuous. Let $x, y \in [0, 1)$ with $f(x) = f(y)$. This means $[x] = [y]$, more precisely:

$$[x] = \{x + k \mid k \in \mathbb{Z}\} = \{y + \ell \mid \ell \in \mathbb{Z}\} = [y] \ .$$

In particular, there has to be $\ell \in \mathbb{Z}$, such that $x = y + \ell$. In other words, we have $x - y \in \mathbb{Z}$. But since $x, y \in [0, 1)$, the difference $x - y$ is in the open interval $(-1, 1)$. Since $(-1, 1) \cap \mathbb{Z} = \{0\}$, we have $x = y$ and f is injective. Let $[x] \in \mathbb{R}/\sim$. Then

$x' = x - \lfloor x \rfloor \in [0,1)$ and $[x] = [x']$. Moreover, $f(x') = [x]$. This proves that f is also surjective and hence bijective.

To see that f is not a homeomorphism, we need to find an open subset $U \subset [0,1)$, such that $f(U)$ is not open in $\mathbb{R}/\sim$. Let $U = [0, \frac{1}{4})$. This is open in $[0,1)$. Observe that

$$q^{-1}(f(U)) = \cdots \cup \left[-1, -\frac{3}{4}\right) \cup \left[0, \frac{1}{4}\right) \cup \left[1, \frac{5}{4}\right) \cup \ldots$$

This is not open in $\mathbb{R}$. In fact, if it were, then there would be an interval of the form $(-\epsilon, \epsilon)$ for some $\epsilon > 0$ in $q^{-1}(f(U))$, since $0 \in q^{-1}(f(U))$. However, this is not the case. By the definition of the quotient topology, this means that $f(U)$ can not be open in $\mathbb{R}/\sim$.