Exercises for the lecture
MA3008 – Algebraic Topology

Solutions of Mock Exam 02 Spring Semester 2017

Exercise 1 (Metric spaces).
Let (X, d) be a metric space.

a) Give the definition of the metric topology $\mathcal{T}(d)$.

b) Let $x \in X$ and let $r > 0$. Show that the subset $B_r(x) = \{x' \in X \mid d(x', x) < r\}$ is open with respect to $\mathcal{T}(d)$.

c) Let $d'(x, y) = \sqrt{d(x, y)}$ for all $x, y \in X$. Show that this defines a new metric on X, i.e. that (X, d') is also a metric space.

d) Let $d' \colon X \times X \to \mathbb{R}$ be the metric from part (c). Prove that $\mathcal{T}(d') = \mathcal{T}(d)$.

Solution: Let $B_r(x) \subset X$ be the subset defined in part b). A subset $U \subset X$ is called open with respect to the metric if for every $x \in U$ there is an $r > 0$ such that $B_r(x) \subset U$. The metric topology $\mathcal{T}(d)$ is defined as the set of all $U \subset X$ that are open with respect to the metric d.
For b) let $y \in B_r(x)$ and let $r' = r - d(x, y)$. Observe that $d(x, y) < r$, hence $r' > 0$. Let $y' \in B_{r'}(y)$. Then we have

$$d(y', x) \leq d(y', y) + d(y, x) < r' + d(y, x) = r' + d(x, y) = r \ .$$

Thus, $y' \in B_r(x)$ and $B_{r'}(y) \subseteq B_r(x)$. Altogether we have proven that for any point $y \in B_r(x)$ there is $r' > 0$, such that $B_{r'}(y) \subseteq B_r(x)$. Hence, $B_r(x)$ is open with respect to the metric d.
To solve c) we have to show that d' has the properties of a metric. Since $d(x, y) \geq 0$ and $x \mapsto \sqrt{x}$ maps $\mathbb{R}_+$ to itself, we have that $d'(x, y) \geq 0$. Observe that

$$d'(x, y) = 0 \qquad \Leftrightarrow \qquad 0 = d'(x, y)^2 = d(x, y) \qquad \Leftrightarrow \qquad x = y \ .$$

Moreover, $d'(x, y) = \sqrt{d(x, y)} = \sqrt{d(y, x)} = d'(y, x)$. Hence, d' is symmetric. To see that the triangle inequality also holds note that

$$\begin{aligned} d'(x,z)^2 &= d(x,z) \le d(x,y) + d(y,z) \\ &\le d(x,y) + d(y,z) + 2\sqrt{d(x,y)}\sqrt{d(y,z)} \\ &= \left(\sqrt{d(x,y)} + \sqrt{d(y,z)}\right)^2 \\ &= \left(d'(x,y) + d'(y,z)\right)^2 \end{aligned}$$

Since the square root function is monotonically increasing, we obtain $d'(x,z) \le d'(x,y) + d'(y,z)$. Thus, d' is indeed a metric.

Concerning part d) of the question let $x \in X$, let $r > 0$ and let $B_r^d(x)$ and $B_r^{d'}(x)$ be the open balls of radius r around x with respect to the metric d or d' respectively. Let $y \in B_r^d(x)$. Since the square root function is monotonically increasing, we have $d'(x,y) = \sqrt{d(x,y)} < \sqrt{r}$ and hence $y \in B_{\sqrt{r}}^{d'}(x)$ and therefore $B_r^d(x) \subseteq B_{\sqrt{r}}^{d'}(x)$. Likewise, let $y \in B_{\sqrt{r}}^{d'}(x)$. The function $x \mapsto x^2$ is monotonically increasing on the non-negative half-line. Hence, we have $d(x,y) = d'(x,y)^2 < r$. This implies $B_{\sqrt{r}}^{d'}(x) \subseteq B_r^d(x)$ and therefore $B_{\sqrt{r}}^{d'}(x) = B_r^d(x)$. Any subset $U \subseteq X$ that is open with respect to the metric d is of the form

$$U = \bigcup_{y \in U} B_{r(y)}^d(y) = \bigcup_{y \in U} B_{\sqrt{r(y)}}^{d'}(y)$$

which shows that it is also open with respect to the metric d'. The same argument proves that $\mathcal{T}(d) = \mathcal{T}(d')$.

Exercise 2 (Connectedness and continuous maps).
Let $\mathbb{R}$ and $\mathbb{R}^2$ be equipped with their standard metric topologies and let $S^1 \subset \mathbb{R}^2$ and $S^0 = \{-1, 1\} \subset \mathbb{R}$ be equipped with their subspace topologies.

a) Give the definitions of path-connected and of connected.

b) Show that the subspace topology $S^0 \subset \mathbb{R}$ is the same as the discrete topology on S^0.

c) Let $f \colon S^1 \to S^0$ be a continuous map. Prove that it has to be equal either to the constant map with value 1 or the constant map with value -1.

d) Suppose that $f \colon S^1 \to \mathbb{R}$ is a continuous map. Prove that there has to be a point $x \in S^1$, such that $f(x) = f(-x)$.

   Hint: Suppose that there is no such point and consider the map

$$g \colon S^1 \to S^0 \quad ; \quad g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|} \ .$$

Solution A topological space $(X, \mathcal{T}_X)$ is path-connected if for any two points $x, y \in X$ there exists a continuous path $\gamma \colon I \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. A topological space $(X, \mathcal{T}_X)$ is disconnected if there are two non-empty open subsets $U, V \subseteq X$ with $U \cap V = \emptyset$ and $U \cup V = X$. The space X is connected if it is not disconnected.
For part b) observe that $J = (-2, 0) \subset \mathbb{R}$ is open in the metric topology on $\mathbb{R}$. Since $\{-1\} = J \cap S^0$, we see that $\{-1\} \subset S^0$ is open in the subspace topology. Likewise, $(0, 2) \subset \mathbb{R}$ is open and therefore $\{1\} = (0, 2) \cap S^0$ is also open. This implies that the subspace topology on S^0 induced by $\mathbb{R}$ is the discrete topology.
Let $U = \{-1\} \subset S^0$ and $V = \{1\} \subset S^0$. These two sets are non-empty and open and we have $S^0 = U \cup V$ and $U \cap V = \emptyset$. Hence, S^0 is disconnected. By a Theorem 4.0.4 in the lecture notes the continuous image of a connected space is connected. For a continuous map $f \colon S^1 \to S^0$ as in part c) we therefore must have $f(S^1) \subset \{-1\}$ or $f(S^1) \subset \{1\}$. But this is the same as saying that f is constant with value either -1 or 1.
To prove d) assume that there is no $x \in S^1$ with the property that $f(x) = f(-x)$. This means that $f(x) - f(-x) \neq 0$ for all $x \in S^1$. The map $g \colon S^1 \to S^0$ with $g(x) = \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$ described in the hint is then continuous. By part c) we have that $g(x)$ is constant with value either -1 or 1. But this is impossible, since

$$g(-x) = \frac{f(-x) - f(x)}{|f(-x) - f(x)|} = -\frac{f(x) - f(-x)}{|f(x) - f(-x)|} = -g(x) \ .$$

We arrived at a contradiction, hence there must be a point $x \in S^1$ with $f(x) = f(-x)$.

Exercise 3 (Compact spaces).

a) Give the definition of compact topological space and the definition of Hausdorff space.

b) Let $(X, \mathcal{T}_X)$ be a compact topological space and let $A \subset X$ be a closed subspace. Prove that A is compact.

c) Let $\mathbb{R}$ be equipped with its standard metric topology. Show that if X is compact and $f\colon X \to \mathbb{R}$ is a continuous map, then f is bounded and takes on a minimum and a maximum value.

d) Let $(X, \mathcal{T}_X)$ be a topological space, let $A \subset X$ and $B \subset X$ be compact subspaces. Show that $A \cup B$ is a compact subspace as well.

Solution A family $(U_i)_{i \in I}$ of open subsets of a topological space $(X, \mathcal{T}_X)$ is called an open cover of X if

$$X = \bigcup_{i \in I} U_i \ .$$

A topological space $(X, \mathcal{T}_X)$ is called compact if for any open cover $(U_i)_{i \in I}$ there is a finite subset $\{i_1, \ldots, i_n\} \subset I$ with the property that

$$X = \bigcup_{k=1}^{n} U_{i_k} = U_{i_1} \cup \cdots \cup U_{i_n} \ .$$

A topological space $(X, \mathcal{T}_X)$ is called a Hausdorff space if for any two points $x, y \in X$ with $x \neq y$ there are open subsets $U, V \subset X$ with $x \in U$, $y \in V$ and $U \cap V = \emptyset$. This solves a). Concerning part b) let $A \subset X$ be a closed subspace of a compact topological space $(X, \mathcal{T}_X)$. Let $(U_i)_{i \in I}$ be an open cover of the subspace A. By the definition of the subspace topology there is an open subset $V_i \subset X$ for each $i \in I$ with the property that $V_i \cap A = U_i$. Observe that $X \setminus A$ is open, since A is closed. Hence,

$$X = \bigcup_{i \in I} V_i \cup (X \setminus A)$$

is an open cover of X. Since X is compact, there is a finite subset $\{i_1, \ldots, i_n\} \subseteq I$ such that

$$X = V_{i_1} \cup \cdots \cup V_{i_n} \cup (X \setminus A) \ .$$

Then we have

$$A = X \cap A = (V_{i_1} \cap A) \cup \cdots \cup (V_{i_n} \cap A) \cup ((X \setminus A) \cap A) = U_{i_1} \cup \cdots \cup U_{i_n}$$

which proves that A is covered by finitely many of the open subsets U_i. Hence, A is compact.
Let $f\colon X \to \mathbb{R}$ be a continuous map as in part c). Since X is compact, its continuous image $f(X) \subset \mathbb{R}$ is compact as well. From the Theorem of Heine and Borel we obtain that $f(X)$ is closed and bounded. In particular, the boundedness implies that

$$s := \sup\{f(x) \mid x \in X\} < \infty \, .$$

To show that f takes on a maximum value suppose that $s \notin f(X)$. Since $f(X)$ is closed, $\mathbb{R} \setminus f(X)$ is open. Thus, there is an $\epsilon > 0$ with $(s-\epsilon, s+\epsilon) \subset (\mathbb{R} \setminus f(X))$. But since $s \in \mathbb{R}$ is the supremum of $f(X)$, we have that $s - \tilde{\epsilon} \in f(X)$ for any $\tilde{\epsilon} > 0$. This is a contradiction since $(s-\epsilon, s) \cap f(X) = \emptyset$. Therefore we must have $s \in f(X)$. To see that f also takes on a minimum value we use the same argument as above on $-f$.
For part d) let $(U_i)_{i \in I}$ be an open cover of $A \cup B$. Observe that $U_i \cap A$ is open in the subspace topology induced by X on A and $(U_i \cap A)_{i \in I}$ is an open cover of A. Likewise, $(U_i \cap B)_{i \in I}$ is an open cover of B. Since A is compact, there are finitely many indices $\{i_1, \ldots, i_n\} \subset I$ such that $A \subset U_{i_1} \cup \cdots \cup U_{i_n}$. Similarly, there are finitely many indices $\{i_{n+1}, \ldots, i_N\} \subset I$ with $B \subset U_{i_{n+1}} \cup \cdots \cup U_{i_N}$. Therefore

$$A \cup B \subset U_{i_1} \cup \cdots \cup U_{i_N}$$

proving that $A \cup B$ is compact.

Exercise 4 (Homotopy equivalence).

a) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces and let $f_1, f_2 \colon X \to Y$ be two continuous maps. Define what is meant by: f_1 is homotopic to f_2.

b) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. Define what it means for a continuous map $f \colon X \to Y$ to be a homotopy equivalence.

c) Let $Y = \mathbb{R}^3 \setminus \{(0,0,z) \in \mathbb{R}^3 \mid z \in \mathbb{R}\} \subset \mathbb{R}^3$ be equipped with the subspace topology, i.e. Y is the complement of the z-axis. Show that the map $f \colon Y \to \mathbb{R}^2 \setminus \{0\}$ given by $f(x,y,z) = (x,y)$ is a homotopy equivalence.

d) Let $Y \subset \mathbb{R}^3$ be the subspace from part c) and let $x_0 = (1,0,0) \in \mathbb{R}^3$. Prove that $\pi_1(Y, x_0) \cong \mathbb{Z}$. You may use without proof that $\pi_1(S^1, z_0) \cong \mathbb{Z}$ for any basepoint $z_0 \in S^1$ as shown in the lecture.

Solution Two continuous maps $f_1, f_2 \colon X \to Y$ between topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are called homotopic if there is a continuous map $H \colon X \times I \to Y$ with the property that for $H_t(x) = H(x,t)$ we have

$$H_0 = f_1 \qquad \text{and} \qquad H_1 = f_2 \ .$$

We denote this by $f_1 \sim_h f_2$.
A continuous map $f \colon X \to Y$ is called a homotopy equivalence if there is a continuous map $g \colon Y \to X$ such that $f \circ g \sim_h \mathrm{id}_Y$ and $g \circ f \sim_h \mathrm{id}_X$.
Let $Y \subset \mathbb{R}^3$ be as in part c), let $X = \mathbb{R}^2 \setminus \{0\}$ and let $f \colon Y \to X$ be the continuous map given by $f(x,y,z) = (x,y)$. Define $g \colon X \to Y$ by $g(x,y) = (x,y,0)$. This is well-defined, since $x \neq 0$ or $y \neq 0$. Observe that

$$f \circ g = \mathrm{id}_X \sim_h \mathrm{id}_X$$

and $(g \circ f)(x,y,z) = (x,y,0)$. Define $H \colon Y \times I \to Y$ by

$$H((x,y,z),t) = (x,y,t \cdot z) \ .$$

Just as above, this map is well-defined, since $x \neq 0$ or $y \neq 0$ implies $(x,y,t \cdot z) \in Y$. Moreover, H is continuous and we have $H_0(x,y,z) = (x,y,0) = (g \circ f)(x,y,z)$ and $H_1 = \mathrm{id}_Y$. Hence, we also have $g \circ f \sim_h \mathrm{id}_Y$, which means that f is a homotopy equivalence.
Let $x_0' = (1,0) \in X$ and note that $f \colon Y \to X$ and $g \colon X \to Y$ are based continuous maps between the pointed topological spaces (Y, x_0) and (X, x_0'). Moreover, we have $H_t(x_0) = x_0$

for all $t \in I$. Therefore H is a based homotopy equivalence between $g \circ f$ and id_Y. Thus, we know that f induces an isomorphism

$$f_*\colon \pi_1(Y, x_0) \to \pi_1(X, x_0')$$

To solve d) it therefore suffices to construct a based homotopy equivalence between (X, x_0') and (S^1, z_0) for some basepoint $z_0 \in S^1$. Let $z_0 = (1, 0)$. The inclusion $f'\colon S^1 \to X$ is a based continuous map. Define $g'\colon X \to S^1$ to be

$$g'(v) = \frac{v}{\|v\|}$$

This is well-defined, since X does not contain the origin. Moreover, it is a based continuous map, because $g'(x_0') = z_0$. We have $g' \circ f' = \mathrm{id}_{S^1}$ and

$$(f' \circ g')(v) = \frac{v}{\|v\|} \ .$$

Define $H'\colon X \times I \to X$ to be

$$H'(v, t) = \left(t + (1 - t)\frac{1}{\|v\|}\right) v \ .$$

This is well-defined, since the term in brackets is never equal to 0. Now note that $H_0' = f' \circ g'$, $H_1' = \mathrm{id}_X$ and $H_t(x_0') = x_0'$, since $\|x_0'\| = 1$. In particular, $f' \circ g' \sim_{h,+} \mathrm{id}_X$. Therefore f' induces an isomorphism

$$f_*'\colon \pi_1(X, x_0') \to \pi_1(S^1, z_0) \ .$$

Thus, $f_*' \circ f_*\colon \pi_1(Y, x_0) \to \pi_1(S^1, z_0)$ is an isomorphism as well. Altogether we obtain

$$\pi_1(Y, x_0) \cong \pi_1(S^1, z_0) \cong \mathbb{Z} \ .$$