Quotient Spaces of the Unit Square

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The 2-sphere

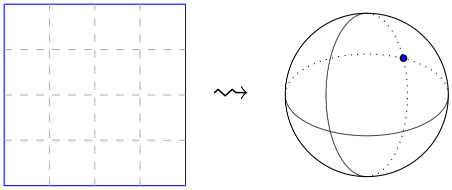
Let $I = [0,1]$ and let $I^2 = [0,1] \times [0,1]$ be the unit square. Let

$$\partial I^2 = (\{0,1\} \times I) \cup (I \times \{0,1\}) \subset I^2$$

be its boundary. Consider the following equivalence relation on I^2:

$$(s_1, t_1) \sim (s_2, t_2) \qquad \Leftrightarrow \qquad (s_1 = s_2 \text{ and } t_1 = t_2)$$
$$\text{or } ((s_1, t_1) \text{ and } (s_2, t_2) \text{ are both in } \partial I^2)$$

This relation collapses the boundary ∂I^2 of the unit square to a single point.

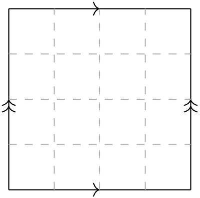


The 2-torus

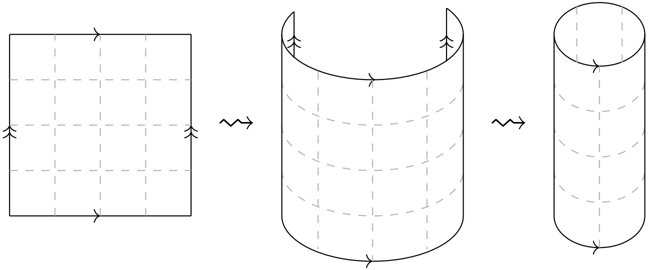
Consider the following equivalence relation on $I^2 = [0,1] \times [0,1]$:

$$
\begin{aligned}
(s_1, t_1) \sim (s_2, t_2) \qquad \Leftrightarrow \qquad & (s_1 = s_2 \text{ and } t_1 = t_2) \\
& \text{or } (s_1, s_2 \in \{0,1\} \text{ and } t_1 = t_2) \\
& \text{or } (s_1 = s_2 \text{ and } t_1, t_2 \in \{0,1\}) \\
& \text{or } (s_1, s_2 \in \{0,1\} \text{ and } t_1, t_2 \in \{0,1\})
\end{aligned}
$$

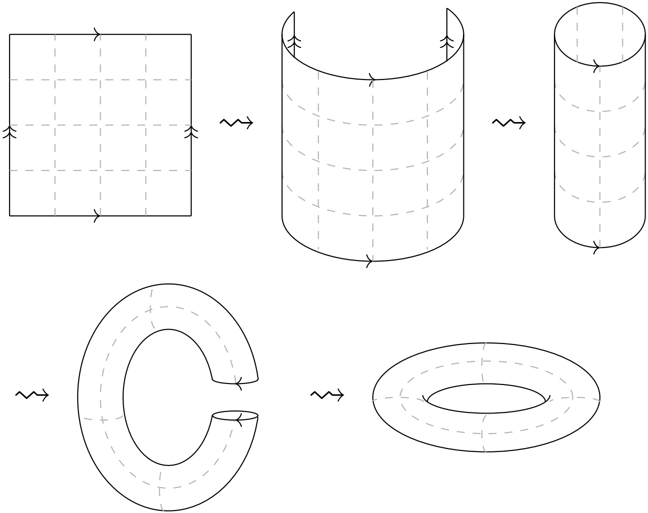
This relation identifies the left edge with the right one in an orientation preserving way and the top one with the bottom one in the same fashion.



The 2-torus



The 2-torus

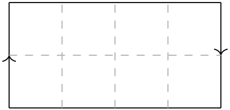


Gluing with a twist

Consider the following equivalence relation on $I^2 = [0,1] \times [0,1]$:

$$(s_1, t_1) \sim (s_2, t_2) \quad \Leftrightarrow \quad \begin{array}{l} (s_1 = s_2 \text{ and } t_1 = t_2) \\ \text{or } (s_1, s_2 \in \{0,1\} \text{ and } t_1 = 1 - t_2) \end{array}$$

This identifies the left edge with the right edge after performing a twist:

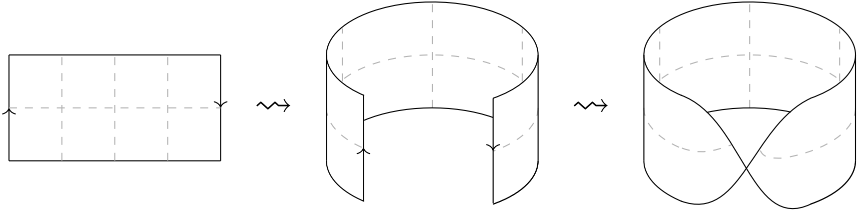


The Möbius band

Consider the following equivalence relation on $I^2 = [0,1] \times [0,1]$:

$$(s_1, t_1) \sim (s_2, t_2) \quad \Leftrightarrow \quad \begin{aligned} &(s_1 = s_2 \text{ and } t_1 = t_2) \\ \text{or } &(s_1, s_2 \in \{0,1\} \text{ and } t_1 = 1 - t_2) \end{aligned}$$

This identifies the left edge with the right edge after performing a twist:
The resulting space $M = I^2/\sim$ is called the **Möbius band**.

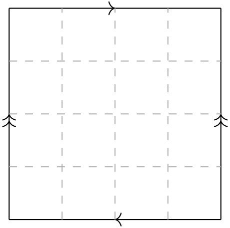


Combining the two relations

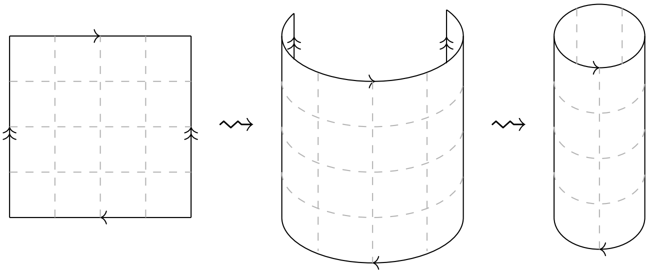
Combining the twisted version with the untwisted one gives an equivalence relation on I^2 of the form:

$$
\begin{aligned}
(s_1, t_1) \sim (s_2, t_2) \qquad \Leftrightarrow \qquad & (s_1 = s_2 \text{ and } t_1 = t_2) \\
\text{or } & (s_1, s_2 \in \{0, 1\} \text{ and } t_1 = t_2) \\
\text{or } & (s_1 = 1 - s_2 \text{ and } t_1, t_2 \in \{0, 1\}) \\
\text{or } & (s_1, s_2 \in \{0, 1\} \text{ and } t_1, t_2 \in \{0, 1\})
\end{aligned}
$$

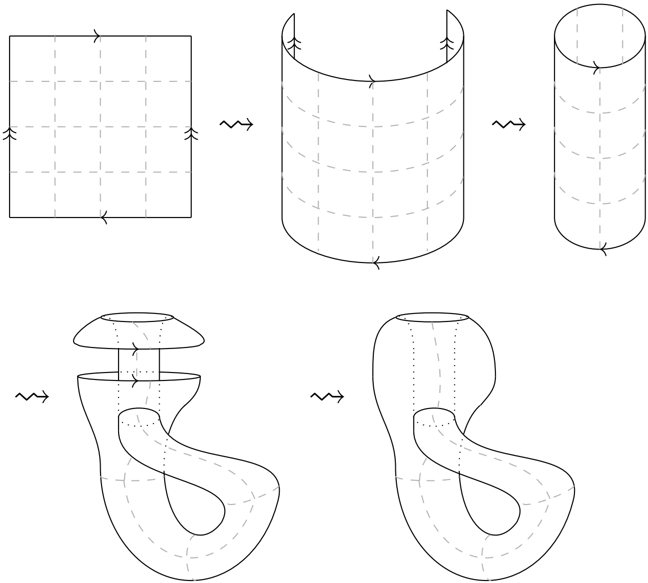
This relation identifies the left edge with the right one preserving its orientation and the top edge with the reversed bottom edge.



The Klein bottle



The Klein bottle



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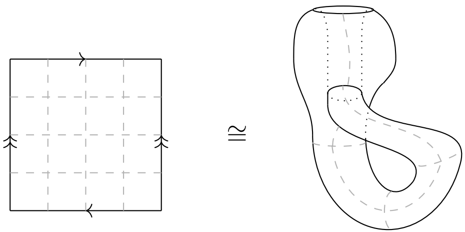
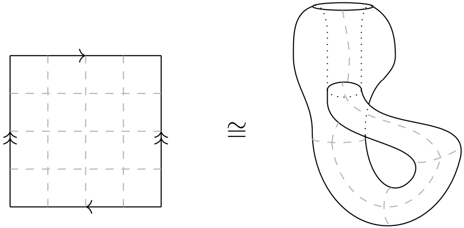
\text{or } & (s_1, s_2 \in \{0, 1\} \text{ and } t_1 = t_2) \\
\text{or } & (s_1 = 1 - s_2 \text{ and } t_1, t_2 \in \{0, 1\}) \\
\text{or } & (s_1, s_2 \in \{0, 1\} \text{ and } t_1, t_2 \in \{0, 1\})
\end{aligned}
$$

The resulting space $K = I^2/\sim$ is called the **Klein bottle**.



The Klein bottle

The resulting space $K = I^2/\sim$ is called the **Klein bottle**.

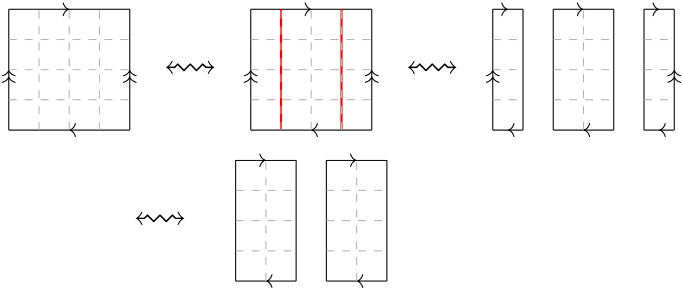


Some facts about K:

- The name is (probably) based on a mistranslation.
- It is not embeddable into $\mathbb{R}^3$, hence the self-intersection in the picture.
- It is, however, embeddable into $\mathbb{R}^4$.
- It does not enclose a volume.
- It has only one side.

The Klein bottle glued from two Möbius bands

Consider the equivalence relation $\sim$ on I^2 as on the last slide. Let $K = I^2/\sim$ be the quotient space, i.e. the Klein bottle. Let us see that it consists of **two Möbius bands glued together along their boundary circle.**



Note that the red line in the second picture is a single circle in K due to the equivalence relation. To get from the last picture on the first line to the picture in the second, we identify the edges with the double arrows. We see that K can also be written as a quotient space of two Möbius bands by an equivalence relation that identifies their boundary circles with one another.