Exercise Sheet 1
Linear Algebra II

Autumn semester 2018

Exercise 1. Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answers.

a) $\mathcal{W}_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2, a_3 = -a_2\}$

b) $\mathcal{W}_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$

c) $\mathcal{W}_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1^2 - 3a_3^2 = 0\}$

Exercise 2. Consider the following definition:
DEFINITION **0.1.** If S_1 and S_2 are nonempty subsets of a vector space $\mathcal{V}$, then the **sum** of S_1 and S_2, denoted by $S_1 + S_2$, is the set $\{x + y : x \in S_1, y \in S_2\}$.

Now consider two subspaces $\mathcal{W}_1$ and $\mathcal{W}_2$ of $\mathcal{V}$.

a) Prove that $\mathcal{W}_1 + \mathcal{W}_2$ is a subspace of $\mathcal{V}$ that contains both $\mathcal{W}_1$ and $\mathcal{W}_2$.

b) Prove that any subspace of $\mathcal{V}$ that contains both $\mathcal{W}_1$ and $\mathcal{W}_2$ must also contain $\mathcal{W}_1 + \mathcal{W}_2$.

Exercise 3. Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then $\text{span}\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices, when using the usual operations of matrix addition and scalar multiplication.

Exercise 4. Show that if S_1 and S_2 are subsets of a vector space $\mathcal{V}$, then $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$.

Exercise 5. Let S_1 and S_2 be subsets of a vector space $\mathcal{V}$. Prove that $\text{span}(S_1 \cap S_2) \subset \text{span}(S_1) \cap \text{span}(S_2)$. Give an example in which $\text{span}(S_1 \cap S_2)$ and $\text{span}(S_1) \cap \text{span}(S_2)$ are equal and one in which they are unequal.

Exercise 6. Give an example of three linearly dependent vectors in $\mathbb{R}^3$ such that none of the three is a multiple of another.

Exercise 7. Let $S = \{u_1, u_2, \dots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \text{span}(\{u_1, \dots, u_k\})$ for some k where $1 \leq k < n$.

We first consider the following two definitions:

DEFINITION **0.2.** Given $A \in M_{n\times n}(F)$, we define the **trace** of A as $tr(A) = \sum_{i=1}^{n} A_{ii}$, which is the sum of the entries along the diagonal of A.

DEFINITION **0.3.** Given $A \in M_{n\times n}(F)$, we say A is **skew-symmetric** if $M^t = -M$.

Exercise 8. The set of all $n \times n$ matrices having trace equal to 0 is a subspace $\mathcal{W}$ of $M_{n\times n}(F)$. Find a basis for $\mathcal{W}$. What is the dimension of $\mathcal{W}$.

Exercise 9. The set of all skew-symmetric $n \times n$ matrices is a subspace $\mathcal{W}$ of $M_{n\times n}(F)$. Find a basis for $\mathcal{W}$. What is the dimension of $\mathcal{W}$?

Exercise 10. Prove that if $\mathcal{W}_1$ and $\mathcal{W}_2$ are finite-dimensional subspaces of a vector space $\mathcal{V}$, then the subspace $\mathcal{W}_1 + \mathcal{W}_2$ is finite-dimensional, and $\dim(\mathcal{W}_1 + \mathcal{W}_2) = \dim(\mathcal{W}_1) + \dim(\mathcal{W}_2) - \dim(\mathcal{W}_1 \cap \mathcal{W}_2)$. (Hint: Consider the vector space $\mathcal{W}_1 \cap \mathcal{W}_2$, its finite basis, and extending it to ones for $\mathcal{W}_1$ and $\mathcal{W}_2$)