Problem sheet for the lecture

MA2008 – Linear Algebra II

Sheet 4 Autumn Semester 2019

Exam Style Questions

Exercise 1 (Determinants).
Compute $\det(A)$, $\det(B)$ and $\det(C)$ for A,B and C given by

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 4 & 4 \end{pmatrix} \quad , \quad C = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 3 \\ 1 & 2 & 3 \end{pmatrix} .$$

Exercise 2 (Cramer's rule).
Consider the following system of linear equations:

$$\begin{aligned} x + y + z &= 2 \\ x - y - z &= 3 \\ x + z &= 4 \end{aligned}$$

a) Use Cramer's rule to solve it.

b) Solve it again using row reduction.

Exercise 3 (Adjugate matrix and inverse).
For the matrix A below, compute the adjugate matrix $\operatorname{adj}(A)$, then compute $\operatorname{adj}(A) \cdot A$, and use this to determine A^{-1}.

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 4 & 0 & 6 \\ 0 & 1 & -1 \end{pmatrix}$$

Exercise 4 (Eigenvectors and eigenvalues).
Consider the matrix $A \in M_{2\times 2}(\mathbb{R})$ given by

$$A = \begin{pmatrix} 1 & -6 \\ 2 & -6 \end{pmatrix}$$

a) Show that the vectors $v_1 = (3, 2)$ and $v_2 = (2, 1)$ are eigenvectors of A and determine the corresponding eigenvalues.

b) Show that $\beta = \{v_1, v_2\}$ is a basis for $\mathbb{R}^2$ and determine $[L_A]_\beta^\beta$.

c) (bonus round) Find another matrix $B \in M_{2\times 2}(\mathbb{R})$ with the property that $A = BDB^{-1}$ where $D = [L_A]_\beta^\beta$. Use this to find a formula for A^k for all $k \in \mathbb{N}_0$.

Intermediate Level Questions

Exercise 5 (Rotation matrices).
For $\theta \in \mathbb{R}$ define the matrix

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} .$$

a) Compute $R(\alpha) \cdot R(\beta)$ and express it in a form analogous to the one of $R(\theta)$.

b) Compute $\det(R(\theta))$ and $R(\theta)^{-1}$.

Exercise 6 (Determinant of a 4×4-matrix).
Compute the determinant $\det(A)$ of the following matrix $A \in M_{4\times 4}(\mathbb{R})$:

$$A = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix}$$

Exercise 7 (Determinants of antidiagonal matrices).
Let $a, b, c, d \in \mathbb{R}$. Find the determinants $\det(A)$ and $\det(B)$ of the *antidiagonal* matrices $A \in M_{3\times 3}(\mathbb{R})$ and $B \in M_{4\times 4}(\mathbb{R})$ given by

$$A = \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix} .$$

What is the determinant of an $n \times n$ antidiagonal matrix with all its antidiagonal entries equal to 2?

Exercise 8 (Eigenvalues of the inverse).
Let V be a finite-dimensional vector space over a field F. Let $T\colon V \to V$ be an isomorphism.

a) Let $\lambda \in F$ be an eigenvalue of T. Show that $\lambda \neq 0$ and $\lambda^{-1} \in F$ is an eigenvalue of the linear transformation T^{-1}.
b) Prove that the eigenspace E_λ of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.
c) Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

Challenges

Exercise 9 (Diagonalising the transposition of 2×2-matrices).
Let $T\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear transformation given by transposition, i.e.

$$T(A) = A^t \ .$$

a) Determine the characteristic polynomial $p_T(t)$ of T and the eigenvalues of T.
b) Is T diagonalizable? If yes, find a basis β for the vector space $M_{2\times 2}(\mathbb{R})$ such that $[T]_\beta^\beta$ is a diagonal matrix.