Problem sheet for the lecture

MA2008 – Linear Algebra II

Sheet 6 Autumn Semester 2019

Exam Style Questions

Exercise 1 (Diagonalisation and determinants).
Suppose that $A \in M_{3\times 3}(\mathbb{R})$ is a matrix with eigenvalues

$$\lambda_1 = -1 \quad , \quad \lambda_2 = 1 \quad \text{and} \quad \lambda_3 = 2 \ .$$

a) Is A diagonalisable? Explain your answer.

b) Let $\alpha = \{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{R}^n$ and let $B \in M_{n\times n}(\mathbb{R})$. Show that

$$[L_B]_\alpha^\alpha = B \ .$$

c) Use a), the result proven in part b) and what you know about determinants of linear transformations to compute $\det(A^5)$, $\det(A^t)$ and $\det(3A)$.

d) What are the eigenvalues of $A + I_3$?

Exercise 2 (Diagonalisation).
Let $T\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation given as follows

$$T(ax^2 + bx + c) = cx^2 + bx + a \ .$$

a) Determine the characteristic polynomial $p_T(t)$ of T and the eigenvalues of T.

b) Find a basis β for the vector space $P_2(\mathbb{R})$ such that $[T]_\beta^\beta$ is a diagonal matrix.

Exercise 3 (Generalised eigenspaces).
Let $L_A\colon \mathbb{C}^4 \to \mathbb{C}^4$ be the linear transformation given by the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Show that the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$ and find the eigenspaces E_2 and E_3 and the generalised eigenspaces $\bar{E}_2$ and $\bar{E}_3$.

Exercise 4 (Diagonalisability).
Let $A \in M_{5\times 5}(\mathbb{C})$ and assume that

$$\lambda_1 = 2 \quad , \quad \lambda_2 = 4 \quad \text{and} \quad \lambda_3 = 6$$

are its only eigenvalues. Suppose that $\dim(E_{\lambda_1}) = 1$, $\dim(E_{\lambda_2}) = 2$ and $\dim(E_{\lambda_3}) = 1$. Is the matrix A diagonalisable? Explain your answer!

Exercise 5 (Division of polynomials).
In each of the following divide the polynomial p by the polynomial b, i.e. find a polynomial q such that $p = q \cdot b + r$ with b as given and a remainder polynomial r with $\deg(r) \leq \deg(b)$.

a) $p(x) = 12x^3 - 11x^2 + 9x + 18$ and $b(x) = 4x + 3$,
b) $p(x) = 3x^4 - 5x^2 + 3$ and $b(x) = x + 2$,
c) $p(x) = 4x^4 + 3x^3 + 2x + 1$ and $b(x) = x^2 + x + 2$.

Intermediate Level Questions

Exercise 6 (Direct Sums).
Let V be a vector space and let $V_1, \ldots, V_k$ be subspaces of V with the property that $V = V_1 \oplus \cdots \oplus V_k$. Show that for any $v \in V$ there are vectors $v_i \in V_i$ for all $i \in \{1, \ldots, k\}$ such that

$$v = v_1 + \cdots + v_k \ .$$

Moreover, prove that this decomposition is unique, i.e. show that if $v_i' \in V_i$ for $i \in \{1, \ldots, k\}$ is another set of vectors with $v = v_1' + \cdots + v_k'$, then $v_i' = v_i$.

Exercise 7 (Jordan normal form).
Let $L_A \colon \mathbb{C}^4 \to \mathbb{C}^4$ be the linear transformation given by the matrix

$$A = \begin{pmatrix} 7 & 1 & -8 & -1 \\ 0 & 3 & 0 & 0 \\ 4 & 2 & -5 & -1 \\ 0 & -4 & 0 & -1 \end{pmatrix}$$

a) Determine the characteristic polynomial $p_A(t)$, the eigenvalues of L_A and their multiplicities.
b) Find a basis for each of the eigenspaces E_λ.
c) Determine a Jordan basis for A, i.e. find a basis β for $\mathbb{C}^4$ with the property that $[L_A]_\beta^\beta$ is in Jordan normal form.

Exercise 8 (Nilpotent linear transformations).

In this exercise we will look at some other properties of nilpotent linear transformations.

a) Find an example of a vector space V and two nilpotent linear transformations $S\colon V \to V$ and $T\colon V \to V$ with the property that $S \circ T$ is no longer nilpotent.

b) Let V be a finite-dimensional vector space and let $S\colon V \to V$ and $T\colon V \to V$ be nilpotent linear transformations with the property $T \circ S = S \circ T$. Prove that $S \circ T$ is in this case also nilpotent.

c) Is there a finite-dimensional vector space V together with two nilpotent linear transformations $S\colon V \to V$ and $T\colon V \to V$ such that $S \circ T$ is invertible?

Challenges

Exercise 9 (Nilpotent linear transformations).

Let $A \in M_{8\times 8}(\mathbb{C})$ be a matrix with the following properties:

- $L_A\colon \mathbb{C}^8 \to \mathbb{C}^8$ is nilpotent,
- $\operatorname{rank}(A) = 5$,
- $\operatorname{rank}(A^2) = 2$.

a) Show that the only possible eigenvalue of A is 0.

b) List all the possible Jordan normal forms (up to permutation of the Jordan blocks) you could get for $[L_A]_\beta^\beta$.