Problem sheet for the lecture

MA2008 – Linear Algebra II

Sheet 7 Autumn Semester 2019

Exam Style Questions

Exercise 1 (Eigenvalues and eigenvectors).
Find two different matrices $A \in M_{2\times 2}(\mathbb{R})$ that have the vector $v = (1,1) \in \mathbb{R}^2$ as an eigenvector with corresponding eigenvalue 2.

Exercise 2 (Jordan normal form).
Let $A \in M_{3\times 3}(\mathbb{C})$ be the following matrix:

$$A = \begin{pmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 5 & 0 & 0 \end{pmatrix}$$

a) Determine the characteristic polynomial $p_A(t)$ and the eigenvalues of A.
b) For each eigenvalue λ of A find a basis for the corresponding eigenspace E_λ.
c) Find a Jordan basis for A and determine its Jordan normal form.

Exercise 3 (Inner products).
Let $V = \mathbb{R}^2$ and define $\langle \cdot\,, \cdot \rangle \colon V \times V \to \mathbb{R}$ by

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \frac{1}{2} \left(3x_1x_2 + y_1x_2 + x_1y_2 + 3y_1y_2\right) \ .$$

Show that this defines an inner product on V.

Exercise 4 (The trace of a matrix).
For a matrix $A \in M_{n\times n}(\mathbb{R})$ with entries $a_{ij} \in \mathbb{R}$ the trace of A is given by

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii} \ .$$

a) Show that $\operatorname{tr}\colon M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ is a linear transformation.
b) Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for any $A, B \in M_{n\times n}(\mathbb{R})$.
c) Show that $\operatorname{tr}(A^tA) \geq 0$ and that $\operatorname{tr}(A^tA) = 0$ if and only if $A = 0$.

Intermediate Level Questions

Exercise 5 (Duals of linear transformations).
Let V and W be finite-dimensional vector spaces over $\mathbb{R}$. Let $T\colon V \to W$ be a linear transformation and let

$$T^*\colon W^* \to V^* \quad , \quad \varphi \mapsto \varphi \circ T \ .$$

a) Show that $T^*(\varphi)$ is indeed an element of V^* for all $\varphi \in W^*$.
b) Show that T^* is a linear transformation.
c) Suppose that $\dim(V) = n$ and $\dim(W) = m$ respectively. Let $\beta = \{v_1, \ldots, v_n\}$ be a basis for V and let $\gamma = \{w_1, \ldots, w_m\}$ be a basis for W. Denote the dual bases by β^* and γ^* respectively. Prove that

$$[T^*]_{\gamma^*}^{\beta^*} = \left([T]_\beta^\gamma\right)^t \ .$$

Exercise 6 (Gram–Schmidt orthonormalisation process).
Consider the vector space $\mathbb{R}^3$ equipped with the dot product as inner product. Use the Gram–Schmidt orthonormalisation process to transform $\{u_1, u_2, u_3\} \subset \mathbb{R}^3$ with

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad , \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad , \quad u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ .$$

into an orthonormal basis.

Challenges

Exercise 7 (Linear forms and kernels).
Let V be a vector space over the field F. Let $\alpha, \beta \in V^*$ be linear forms on V with the property that $\alpha(v) = 0$ if and only if $\beta(v) = 0$. Show that in this situation we have $\alpha = \lambda \beta$ for some $\lambda \in F$.