Solutions to the problem sheet for

MA2008 – Linear Algebra II

Sheet 4 Autumn Semester 2019

Exercise 1 (Determinants).
Compute $\det(A)$, $\det(B)$ and $\det(C)$ for A,B and C given by

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 4 & 4 \end{pmatrix} \quad , \quad C = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 3 \\ 1 & 2 & 3 \end{pmatrix} \ .$$

Solution to 1: We can compute $\det(A)$ according to the rule for 2×2-matrices, i.e.

$$\det(A) = 1 \cdot 4 - 2 \cdot 2 = 0 \ .$$

Another way to see this directly is by noticing that the second row is a multiple of the first one. Therefore the rows are linearly dependent and the determinant has to vanish. To compute $\det(B)$ we may use the cofactor expansion along the first column. This gives

$$\det(B) = 1 \cdot \det \begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix} + 1 \cdot \det \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} = 1 \cdot 0 + 1 \cdot (2 \cdot 1 - 1 \cdot 3) = -1 \ .$$

Finally, to compute $\det(C)$ we can first subtract two times the last row from the first and the second. This does not change the determinant and we obtain

$$\det(C) = \det \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \det \begin{pmatrix} 0 & -2 & -4 \\ 0 & 0 & -3 \\ 1 & 2 & 3 \end{pmatrix}$$

Now a cofactor expansion along the first column gives

$$\det(C) = 1 \cdot \det \begin{pmatrix} -2 & -4 \\ 0 & -3 \end{pmatrix} = (-2) \cdot (-3) - 0 \cdot (-4) = 6 \ .$$

Exercise 2 (Cramer's rule).
Consider the following system of linear equations:

$$\begin{aligned} x + y + z &= 2 \\ x - y - z &= 3 \\ x + z &= 4 \end{aligned}$$

a) Use Cramer's rule to solve it.

b) Solve it again using row reduction.

Solution to 2 a) The matrix $A \in M_{3\times3}(\mathbb{R})$ of the given system is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

The system can be written in matrix form as $Ax = b$ with

$$b = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

To compute the determinant of A we can use the first row to eliminate the two (-1)'s in the second row and use cofactor expansion along the second column:

$$\det(A) = \det \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = -\det \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = -2 \ .$$

If we denote the components of the solution vector by $x = (x_1, x_2, x_3)$, then by Cramer's rule:

$$x_1 = -\frac{1}{2} \det \begin{pmatrix} 2 & 1 & 1 \\ 3 & -1 & -1 \\ 4 & 0 & 1 \end{pmatrix} = -\frac{1}{2} \det \begin{pmatrix} 2 & 1 & 1 \\ 5 & 0 & 0 \\ 4 & 0 & 1 \end{pmatrix} = \frac{1}{2} \det \begin{pmatrix} 5 & 0 \\ 4 & 1 \end{pmatrix} = \frac{5}{2} \ ,$$

where we used the first row to eliminate the (-1)'s in the second row and then the cofactor expansion along the second column. The second component yields

$$x_2 = -\frac{1}{2} \det \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & -1 \\ 1 & 4 & 1 \end{pmatrix} = -\frac{1}{2} \det \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 2 & 0 \end{pmatrix} = -\frac{1}{2} \det \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix} = -2 \ ,$$

where we used the first row to eliminate the leading ones in the second and the third row, followed by cofactor expansion along the first column. The third component gives

$$x_3 = -\frac{1}{2}\det\begin{pmatrix}1 & 1 & 2\\ 1 & -1 & 3\\ 1 & 0 & 4\end{pmatrix} = -\frac{1}{2}\det\begin{pmatrix}1 & 1 & 2\\ 2 & 0 & 5\\ 1 & 0 & 4\end{pmatrix} = \frac{1}{2}\det\begin{pmatrix}2 & 5\\ 1 & 4\end{pmatrix} = \frac{3}{2}\,,$$

where we used the first row to eliminate the -1 in the second row, followed by cofactor expansion along the second column.

Solution to 2 b) We can also solve this system of linear equations using row reduction. The augmented matrix $(A|b)$ takes the following form:

$$(A|b) = \begin{pmatrix}1 & 1 & 1 & 2\\ 1 & -1 & -1 & 3\\ 1 & 0 & 1 & 4\end{pmatrix}$$

We can subtract the first row from the second and the third and divide the second row by (-2) to arrive at the following result

$$\begin{pmatrix}1 & 1 & 1 & 2\\ 0 & 1 & 1 & -\frac{1}{2}\\ 0 & -1 & 0 & 2\end{pmatrix}$$

Adding the second row to the third and subtracting it from the first yields:

$$\begin{pmatrix}1 & 0 & 0 & \frac{5}{2}\\ 0 & 1 & 1 & -\frac{1}{2}\\ 0 & 0 & 1 & \frac{3}{2}\end{pmatrix}$$

Finally, subtracting the third row from the first brings us to:

$$\begin{pmatrix}1 & 0 & 0 & \frac{5}{2}\\ 0 & 1 & 0 & -2\\ 0 & 0 & 1 & \frac{3}{2}\end{pmatrix}$$

As we see from this matrix we get the same solution vector $(\frac{5}{2}, -2, \frac{3}{2})$, but the method via row reduction is much simpler.

Exercise 3 (Adjugate matrix and inverse).
For the matrix A below, compute the adjugate matrix $\mathrm{adj}(A)$, then compute $\mathrm{adj}(A) \cdot A$, and use this to determine A^{-1}.

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 4 & 0 & 6 \\ 0 & 1 & -1 \end{pmatrix}$$

Solution to 3: The cofactors of the matrix A are computed as follows:

$$\begin{aligned}
c_{11} &= \det \begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix} = -6 \\
c_{12} &= -\det \begin{pmatrix} 4 & 6 \\ 0 & -1 \end{pmatrix} = 4 \\
c_{13} &= \det \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} = 4 \\
c_{21} &= -\det \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = 1 \\
c_{22} &= \det \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = -1 \\
c_{23} &= -\det \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = -1 \\
c_{31} &= \det \begin{pmatrix} -1 & 2 \\ 0 & 6 \end{pmatrix} = -6 \\
c_{32} &= -\det \begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix} = 2 \\
c_{33} &= \det \begin{pmatrix} 1 & -1 \\ 4 & 0 \end{pmatrix} = 4
\end{aligned}$$

Therefore the adjugate matrix of A is given by

$$\mathrm{adj}(A) = \begin{pmatrix} -6 & 1 & -6 \\ 4 & -1 & 2 \\ 4 & -1 & 4 \end{pmatrix} .$$

If we now compute the product with A we get

$$\operatorname{adj}(A) \cdot A = \begin{pmatrix} -6 & 1 & -6 \\ 4 & -1 & 2 \\ 4 & -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 2 \\ 4 & 0 & 6 \\ 0 & 1 & -1 \end{pmatrix} = (-2) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore $\det(A) = -2$ and the inverse of A is

$$A^{-1} = \begin{pmatrix} 3 & -\frac{1}{2} & 3 \\ -2 & \frac{1}{2} & -1 \\ -2 & \frac{1}{2} & -2 \end{pmatrix} .$$

Exercise 4 (Eigenvectors and eigenvalues).
Consider the matrix $A \in M_{2\times 2}(\mathbb{R})$ given by

$$A = \begin{pmatrix} 1 & -6 \\ 2 & -6 \end{pmatrix}$$

a) Show that the vectors $v_1 = (3, 2)$ and $v_2 = (2, 1)$ are eigenvectors of A and determine the corresponding eigenvalues.
b) Show that $\beta = \{v_1, v_2\}$ is a basis for $\mathbb{R}^2$ and determine $[L_A]_\beta^\beta$.
c) (bonus round) Find another matrix $B \in M_{2\times 2}(\mathbb{R})$ with the property that $A = BDB^{-1}$ where $D = [L_A]_\beta^\beta$. Use this to find a formula for A^k for all $k \in \mathbb{N}_0$.

Solution to 4 a) To see that $v_1 = (3, 2)$ is an eigenvector of A note that

$$Av_1 = \begin{pmatrix} 1 & -6 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -9 \\ -6 \end{pmatrix} = (-3) \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Therefore v_1 is an eigenvector of A with respect to the eigenvalue (-3).
To see that $v_2 = (2, 1)$ is an eigenvector of A note that

$$Av_1 = \begin{pmatrix} 1 & -6 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} = (-2) \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Therefore v_2 is an eigenvector of A with respect to the eigenvalue (-2).

Solution to 4 b) Since $\beta = \{v_1, v_2\}$ consists of just two vectors, we need to check that they are not multiples of one another, which follows from the fact that

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = a \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

has no solution. This implies that β is a basis for $\mathbb{R}^2$. The matrix corresponding to this basis is easy to determine:

$$[L_A]_\beta^\beta = \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix}$$

Solution to 4 c) Let $D = [L_A]_\beta^\beta$ be the diagonal matrix from part b). Let $\alpha = \{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$. By a theorem from the lecture we have

$$A = [L_A]_\alpha^\alpha = [I_{\mathbb{R}^2}]_\beta^\alpha \cdot [L_A]_\beta^\beta \cdot [I_{\mathbb{R}^2}]_\alpha^\beta \ ,$$

where $I_{\mathbb{R}^2}$ denotes the identity transformation on $\mathbb{R}^2$. Since

$$v_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3e_1 + 2e_2 \quad , \quad v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2e_1 + e_2 \ ,$$

we obtain the matrix

$$[I_{\mathbb{R}^2}]^\alpha_\beta = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \ .$$

We can construct $[I_{\mathbb{R}^2}]^\beta_\alpha$ in a similar way:

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -v_1 + 2v_2 \quad , \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2v_1 - 3v_2 \ ,$$

which gives the matrix

$$[I_{\mathbb{R}^2}]^\beta_\alpha = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix} \ .$$

Note that $[I_{\mathbb{R}^2}]^\beta_\alpha$ and $[I_{\mathbb{R}^2}]^\alpha_\beta$ are inverse to one another. This is of course not a coincidence and we leave it as another exercise to work out why. Let

$$B = [I_{\mathbb{R}^2}]^\alpha_\beta = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

Then we have $A = BDB^{-1}$. Note that

$$\begin{aligned} A^2 &= (BDB^{-1})(BDB^{-1}) = BD^2B^{-1} \\ A^3 &= (BDB^{-1})(BDB^{-1})(BDB^{-1}) = BD^3B^{-1} \\ &\vdots \\ A^k &= BD^kB^{-1} \end{aligned}$$

The last line allows us to write down a formula for A^k:

$$A^k = \begin{pmatrix} (-3)\cdot(-3)^k + 4\cdot(-2)^k & 6\cdot(-3)^k - 6\cdot(-2)^k \\ (-2)\cdot(-3)^k + 2\cdot(-2)^k & 4\cdot(-3)^k - 3\cdot(-2)^k \end{pmatrix}$$

Exercise 5 (Rotation matrices).
For $\theta \in \mathbb{R}$ define the matrix

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} .$$

a) Compute $R(\alpha) \cdot R(\beta)$ and express it in a form analogous to the one of $R(\theta)$.
b) Compute $\det(R(\theta))$ and $R(\theta)^{-1}$.

Solution to 5 a) Recall the following trigonometric identities:

$$\begin{aligned} \cos\alpha\cos\beta - \sin\alpha\sin\beta &= \cos(\alpha+\beta) \\ \cos\alpha\sin\beta + \sin\alpha\cos\beta &= \sin(\alpha+\beta) \\ \sin^2\alpha + \cos^2\alpha &= 1 \\ \cos(-\theta) &= \cos(\theta) \\ \sin(-\theta) &= -\sin(\theta) \end{aligned}$$

We will use them throughout this exercise. Now note that

$$\begin{aligned} R(\alpha) \cdot R(\beta) &= \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \cdot \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} \\ &= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -(\cos\alpha\sin\beta + \sin\alpha\cos\beta) \\ \cos\alpha\sin\beta + \sin\alpha\cos\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} = R(\alpha+\beta) . \end{aligned}$$

Solution to 5 b)

$$\det(R(\theta)) = \cos^2(\theta) + \sin^2(\theta) = 1$$

Moreover, the inverse matrix is given by

$$R(\theta)^{-1} = \frac{1}{\det(R(\theta))} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = R(-\theta) .$$

Exercise 6 (Determinant of a 4×4-matrix).
Compute the determinant $\det(A)$ of the following matrix $A \in M_{4\times 4}(\mathbb{R})$:

$$A = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix}$$

Solution to 6: In a first step we can use the first row to eliminate the leading ones in the second and the last row. This does not change the determinant:

$$\det(A) = \det \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 3 & 2 \end{pmatrix}$$

Using cofactor expansion along the first column yields:

$$\det(A) = \det \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix}$$

We can now use the second row to eliminate the leading one in the third row:

$$\det(A) = \det \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

and use cofactor expansion along the first column again (note the minus sign!) to obtain

$$\det(A) = -\det \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} = -(-2) = 2 \; .$$

Exercise 7 (Determinants of antidiagonal matrices).
Let $a, b, c, d \in \mathbb{R}$. Find the determinants $\det(A)$ and $\det(B)$ of the *antidiagonal* matrices $A \in M_{3\times 3}(\mathbb{R})$ and $B \in M_{4\times 4}(\mathbb{R})$ given by

$$A = \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix} .$$

What is the determinant of an $n \times n$ antidiagonal matrix with all its antidiagonal entries equal to 2?

Solution to 7: To compute the determinant of A we switch the first and the third row and get a minus sign, i.e.

$$\det(A) = \det \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix} = -\det \begin{pmatrix} c & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a \end{pmatrix}$$

This is a diagonal matrix. The determinant of a diagonal matrix is the product of the diagonal entries. Therefore

$$\det(A) = -c \cdot b \cdot a = -abc$$

To compute the determinant of B we have to switch the first with the last row and the second with the third row. This yields

$$\det(B) = \det \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix} = (-1)^2 \det \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

We end up with a diagonal matrix again. Thus,

$$\det(B) = abcd .$$

A matrix with 2's on the antidiagonal looks like this

$$C = \begin{pmatrix} 0 & \cdots & 0 & 2 \\ 0 & \cdots & 2 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 2 & \cdots & 0 & 0 \end{pmatrix}$$

Its determinant can be computed as follows: To bring this into diagonal form we have to switch the first row with the last row, the second row with the second to last row and so on. If $C \in M_{n \times n}(\mathbb{R})$ with $n = 2k$ or $n = 2k + 1$, then this process yields in total k row switches. In particular, note that in case $n = 2k + 1$ is odd, then the middle row does not need to be paired off with any other row. Therefore the determinant is

$$\det(C) = (-1)^k \cdot 2^n \ .$$

Exercise 8 (Eigenvalues of the inverse).
Let V be a finite-dimensional vector space over a field F. Let $T\colon V \to V$ be an isomorphism.

a) Let $\lambda \in F$ be an eigenvalue of T. Show that $\lambda \neq 0$ and $\lambda^{-1} \in F$ is an eigenvalue of the linear transformation T^{-1}.
b) Prove that the eigenspace E_λ of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.
c) Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

Solution to 8 a) Suppose that $\lambda \in F$ is an eigenvalue of T. This means that there is a nonzero vector $v \in V$ with the property that $T(v) = \lambda v$. Suppose $\lambda = 0$. Then v would be a nonzero vector in the null space $\ker(T)$ of T, which would imply that T is not injective. This can not be, since T is an isomorphism by assumption. Therefore, $\lambda \neq 0$. Moreover, note that $v = \lambda^{-1}T(v)$ and hence

$$T^{-1}(v) = T^{-1}(\lambda^{-1}T(v)) = \lambda^{-1}(T^{-1}T(v)) = \lambda^{-1}v\ .$$

In particular, $v \in V$ is an eigenvector of T^{-1} with corresponding eigenvalue λ^{-1}. Thus, we have shown that, if $\lambda \in F$ is an eigenvalue of the isomorphism T, then $\lambda \neq 0$ and λ^{-1} is an eigenvalue of T^{-1}.

Solution to 8 b) Denote the eigenspace of T corresponding to the eigenvalue $\lambda \in F$ by E_λ^T. By definition

$$\begin{aligned} E_\lambda^T &= \{v \in V \mid (T - \lambda I_V)(v) = 0\} \\ &= \{v \in V \mid (I_V - \lambda T^{-1})(v) = 0\} \\ &= \{v \in V \mid (\lambda^{-1} I_V - T^{-1})(v) = 0\} \\ &= \{v \in V \mid (T^{-1} - \lambda^{-1} I_V)(v) = 0\} = E_{\lambda^{-1}}^{T^{-1}}\ . \end{aligned}$$

To obtain the second equality we applied the linear transformation T^{-1} to both sides of the equation $(T - \lambda I_V)(v) = 0$. Likewise, we get the third equality by multiplying both sides of the equation by λ^{-1} and the last equality by multiplying by (-1).

Solution to 8 c) Suppose that T is diagonalizable and let β be a basis for V such that $D = [T]_\beta^\beta$ is a diagonal matrix. Since T is an isomorphism, D has to be invertible by Thm. 3.0.9 and by the same theorem

$$D^{-1} = \left[T^{-1}\right]_\beta^\beta\ .$$

But if D is a diagonal matrix, then so is D^{-1}. In fact, if λ is an entry on the diagonal of D, then $\lambda \neq 0$ (because D is invertible) and λ^{-1} is the corresponding entry in D^{-1}. But this implies that T^{-1} is diagonalizable as well. The converse statement is proven by applying the same argument with the roles of T and T^{-1} interchanged.

Exercise 9 (Diagonalising the transposition of 2×2-matrices).
Let $T\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear transformation given by transposition, i.e.

$$T(A) = A^t \ .$$

a) Determine the characteristic polynomial $p_T(t)$ of T and the eigenvalues of T.
b) Is T diagonalizable? If yes, find a basis β for the vector space $M_{2\times 2}(\mathbb{R})$ such that $[T]_\beta^\beta$ is a diagonal matrix.

Solution to 9 a) In this part of the exercise you are asked to determine the characteristic polynomial of the linear transformation

$$T\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R}) \quad , \quad T(A) = A^t$$

given by transposition. A basis for the vector space $M_{2\times 2}(\mathbb{R})$ over $\mathbb{R}$ is given by the set $\alpha = \{E_{11}, E_{12}, E_{21}, E_{22}\}$, where $E_{ij} \in M_{2\times 2}(\mathbb{R})$ is the matrix, which has only one nonzero entry equal to 1 in the ith row, jth column, i.e.

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

The vector space $M_{2\times 2}(\mathbb{R})$ is four-dimensional. Therefore the matrix representation $[T]_\alpha^\alpha$ is a 4×4-matrix. Observe that $T(E_{ij}) = E_{ji}$. Therefore

$$\begin{aligned}
T(E_{11}) &= E_{11} = 1 \cdot E_{11} + 0 \cdot E_{12} + 0 \cdot E_{21} + 0 \cdot E_{22} \ , \\
T(E_{12}) &= E_{21} = 0 \cdot E_{11} + 0 \cdot E_{12} + 1 \cdot E_{21} + 0 \cdot E_{22} \ , \\
T(E_{21}) &= E_{12} = 0 \cdot E_{11} + 1 \cdot E_{12} + 0 \cdot E_{21} + 0 \cdot E_{22} \ , \\
T(E_{22}) &= E_{22} = 0 \cdot E_{11} + 0 \cdot E_{12} + 0 \cdot E_{21} + 1 \cdot E_{22} \ .
\end{aligned}$$

and the matrix representation of T with respect to α is

$$[T]_\alpha^\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ .$$

We are now able to determine the characteristic polynomial $p_T(t)$:

$$p_T(t) = \det \left(\begin{pmatrix} 1-t & 0 & 0 & 0 \\ 0 & -t & 1 & 0 \\ 0 & 1 & -t & 0 \\ 0 & 0 & 0 & 1-t \end{pmatrix} \right) = -(1-t)^3(1+t) \ ,$$

where we used cofactor expansion with respect to the first and last column to compute the determinant. Thus, the eigenvalues of T are $\lambda_1 = 1$ and $\lambda_2 = -1$.

Solution to 9 b) The eigenspaces E_1 and E_{-1} are defined as follows:

$$\begin{aligned} E_1 &= \ker(T - I_{M_{2\times 2}(\mathbb{R})}) = \left\{A \in M_{2\times 2}(\mathbb{R}) \mid A^t = A\right\} \ , \\ E_{-1} &= \ker(T + I_{M_{2\times 2}(\mathbb{R})}) = \left\{A \in M_{2\times 2}(\mathbb{R}) \mid A^t = -A\right\} \ . \end{aligned}$$

In particular, E_1 is the linear subspace of all symmetric 2×2-matrices over $\mathbb{R}$, i.e.

$$E_1 = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

A basis for this vector space is given by $\{E_{11}, E_{22}, F_1\}$ with E_{11} and E_{22} as above and

$$F_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ .$$

Likewise, E_{-1} is the vector space of antisymmetric 2×2-matrices, i.e.

$$\begin{aligned} E_{-1} &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a = -a,\ c = -b,\ d = -d \right\} \\ &= \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \mid b \in \mathbb{R} \right\} = \left\{ b \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mid b \in \mathbb{R} \right\} \end{aligned}$$

This is a one-dimensional vector space and $\{F_2\}$ with

$$F_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is a basis for E_{-1}. Therefore $\beta = \{E_{11}, E_{22}, F_1, F_2\}$ is a basis for $M_{2\times 2}(\mathbb{R})$ with respect to which the matrix representation of T takes the following form:

$$[T]_\beta^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

As we see, this is indeed a diagonal matrix.