Solutions to the problem sheet for

MA2008 – Linear Algebra II

Sheet 5 Autumn Semester 2019

Exercise 1 (Vectors and matrices).
Compute each of the following matrix-vector products (the last one has entries in $\mathbb{C}$, the first two in $\mathbb{R}$):

$$\begin{pmatrix} 1 & 2 \\ -3 & -4 \\ 5 & -6 \end{pmatrix} \begin{pmatrix} -2 \\ 7 \end{pmatrix} \quad , \quad \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ -3 \\ 4 \end{pmatrix} \quad , \quad \begin{pmatrix} 1 & 1+i & 4 \\ -i & -2 & 3-2i \end{pmatrix} \begin{pmatrix} 2+i \\ -i \\ 1 \end{pmatrix}$$

Solution to 1: The resulting matrix-vector products are:

$$\begin{pmatrix} 12 \\ -22 \\ -52 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} \quad , \quad \begin{pmatrix} 7 \\ 4-2i \end{pmatrix} \; .$$

Exercise 2 (Matrix products).
Let $A, B \in M_{3\times3}(\mathbb{R})$, $C \in M_{4\times3}(\mathbb{R})$ be the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad C = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Compute the following matrix products:

$$A \cdot B \quad , \quad B \cdot A \quad , \quad C \cdot A \quad , \quad A \cdot C^t \quad , \quad C \cdot A^t \ .$$

Solution to 2:

$$A \cdot B = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 2 & 1 \\ 1 & 2 & -2 \end{pmatrix} \quad , \quad B \cdot A = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 2 & -2 \\ 1 & 0 & -1 \end{pmatrix} \quad , \quad C \cdot A = \begin{pmatrix} 4 & 2 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$A \cdot C^t = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ -2 & 1 & -1 & 1 \end{pmatrix} \quad , \quad C \cdot A^t = (A \cdot C^t)^t = \begin{pmatrix} 4 & 2 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

Exercise 3 (Eigenvalues and eigenvectors).
Find all eigenvalues and eigenvectors of each of the following matrices over $\mathbb{C}$:

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \quad , \quad \begin{pmatrix} i & 2+i \\ 0 & 1 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Solution to 3: [In the lecture I solved this questions for the first and the third matrix. I realised after the lecture that I mistakenly considered both matrices over $\mathbb{R}$ not over $\mathbb{C}$ (as stated in the question). Fortunately, this is easy to repair: Every occurrence of $\mathbb{R}^k$ in the lecture just needs to be replaced by $\mathbb{C}^k$. The solution shown below should be correct.]
First consider

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \ .$$

The characteristic polynomial $p_A(t)$ is given by

$$p_A(t) = (1-t)(6-t) - 6 = t^2 - 7t = t(t-7) \ .$$

Hence, the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 7$. To find the eigenvectors corresponding to λ_1 we have to solve

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which gives the linear equation $x + 2y = 0$. Hence, the eigenspace is given by

$$E_{\lambda_1} = \left\{ a \begin{pmatrix} -2 \\ 1 \end{pmatrix} \in \mathbb{C}^2 \mid a \in \mathbb{C} \right\} \ .$$

Any non-zero vector in this space is an eigenvector corresponding to the eigenvalue 0.
To find the eigenvectors corresponding to λ_2 we need to solve the linear equation

$$\begin{pmatrix} -6 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which boils down to $3x - y = 0$ and has the solution set

$$E_{\lambda_2} = \left\{ a \begin{pmatrix} 1 \\ 3 \end{pmatrix} \in \mathbb{C}^2 \mid a \in \mathbb{C} \right\}$$

Any non-zero vector in this space is an eigenvector corresponding to the eigenvalue λ_2.

Next we consider the matrix

$$B = \begin{pmatrix} i & 2+i \\ 0 & 1 \end{pmatrix}$$

Its characteristic polynomial is given by

$$p_B(t) = (i-t)(1-t) \ .$$

Hence, the eigenvalues of B are $\lambda_1 = i$ and $\lambda_2 = 1$. To find the eigenvectors corresponding to λ_1 we need to find the solution set of the linear equation

$$\begin{pmatrix} 0 & 2+i \\ 0 & 1-i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which gives $y = 0$. Therefore the eigenspace of λ_1 is

$$E_{\lambda_1} = \left\{ a \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2 \mid a \in \mathbb{C} \right\}$$

and any non-zero element of E_{λ_1} is an eigenvector corresponding to λ_1. The eigenspace of λ_2 is the solution set of the following linear equation

$$\begin{pmatrix} i-1 & 2+i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ .$$

Note that $(i-1)^{-1} = -\frac{1}{2}(1+i)$. Therefore the eigenspace corresponding to λ_2 is given by

$$\begin{aligned} E_{\lambda_2} &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \mid x = \frac{1}{2}(1+i)(2+i)y = \frac{1}{2}(1+3i)y \right\} \\ &= \left\{ a \begin{pmatrix} 1+3i \\ 2 \end{pmatrix} \in \mathbb{C}^2 \mid a \in \mathbb{C} \right\} \ . \end{aligned}$$

Any non-zero vector in this space is an eigenvector of B corresponding to λ_2.
Finally, consider

$$C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of C is given by

$$\begin{aligned} p_C(t) &= \det \begin{pmatrix} -t & 1 & 1 \\ 1 & -t & 1 \\ 1 & 1 & -t \end{pmatrix} = -t(t^2-1) - (-t-1) + 1 + t \\ &= -(t+1)^2(t-2) \end{aligned}$$

Therefore the (distinct) eigenvalues of C are $\lambda_1 = -1$ and $\lambda_2 = 2$. The eigenspace of λ_1 is the solution set of the linear equation

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which gives

$$E_{\lambda_1} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{C}^3 \mid x + y + z = 0 \right\} = \left\{ a \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \in \mathbb{C}^3 \mid a, b \in \mathbb{C} \right\} .$$

Note that this eigenspace is two-dimensional, but still any non-zero vector in E_{λ_1} is an eigenvector corresponding to λ_1.

The eigenspace of C corresponding to the eigenvalue λ_2 is the solution set of the linear equation

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and is therefore given by

$$E_{\lambda_2} = \left\{ a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{C}^3 \mid a \in \mathbb{C} \right\} .$$

Exercise 4 (Eigenvectors and geometry).
Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(x,y) = (y,x)$$

Let $\alpha = \{e_1, e_2\}$ be the standard basis.

a) Write down the matrix representation $A = [T]_\alpha^\alpha$ for T with respect to α.

b) Show that the eigenvalues of T are $\lambda_1 = 1$ and $\lambda_2 = -1$ and compute corresponding eigenvectors.

c) Find a geometric interpretation for the eigenvectors of T.

Solution to 4 a) The matrix representation is given by

$$[T]_\alpha^\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Solution to 4 b) The characteristic polynomial of T is given by

$$p_T(t) = t^2 - 1 = (t+1)(t-1) \ .$$

Therefore the only eigenvalues of T are $\lambda_1 = 1$ and $\lambda_2 = -1$. To compute the eigenvectors corresponding to $\lambda_1 = 1$ we have to solve the following system of linear equations:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ ,$$

which has the solution set

$$E_{\lambda_1} = \left\{ a \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbb{R}^2 \mid a \in \mathbb{R} \right\} \ .$$

Similarly, we can compute E_{λ_2} by solving the following system of linear equations:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ ,$$

which has the solution set

$$E_{\lambda_2} = \left\{ a \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \mathbb{R}^2 \mid a \in \mathbb{R} \right\} \ .$$

Solution to 4 c) Note that E_{λ_1} is a line through the origin and the point $(1, 1)$ in the euclidean plane. Likewise, E_{λ_2} is a line through the origin and the point $(1, -1)$. The linear transformation T maps both of these lines to itself. However, a vector $x \in E_{\lambda_1}$ is mapped to x, while each vector $y \in E_{\lambda_2}$ is mapped to $-y$, because $\lambda_2 = -1$. Therefore T is the reflection of the plane along the line perpendicular to $(1, -1)$, which is E_{λ_1}.