Solutions to the problem sheet for

MA2008 – Linear Algebra II

Sheet 6 Autumn Semester 2019

Exercise 1 (Diagonalisation and determinants).
Suppose that $A \in M_{3\times 3}(\mathbb{R})$ is a matrix with eigenvalues

$$\lambda_1 = -1 \quad , \quad \lambda_2 = 1 \quad \text{and} \quad \lambda_3 = 2 \ .$$

a) Is A diagonalisable? Explain your answer.

b) Let $\alpha = \{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{R}^n$ and let $B \in M_{n\times n}(\mathbb{R})$. Show that

$$[L_B]_\alpha^\alpha = B \ .$$

c) Use a), the result proven in part b) and what you know about determinants of linear transformations to compute $\det(A^5)$, $\det(A^t)$ and $\det(3A)$.

d) What are the eigenvalues of $A + I_3$?

Solution to 1a) The matrix A is diagonalisable by Cor. 6 in Lec. 8, since the linear transformation $L_A \colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $L_A(v) = Av$ has three distinct eigenvalues and $\dim(\mathbb{R}^3) = 3$.

Solution to 1b) Let $b_{ij} \in \mathbb{R}$ be the entries of the matrix B. Note that

$$Be_j = \sum_{i=1} b_{ij} e_i \ .$$

Therefore the jth column of $[L_B]_\alpha^\alpha$ has the entries $(b_{1j}, \dots, b_{nj})$. But this is just the jth column of B itself. Therefore all columns of $[L_B]_\alpha^\alpha$ agree with the ones of B and hence

$$[L_B]_\alpha^\alpha = B \ .$$

Solution to 1c) We will first compute $\det(A)$. By 1b) we have

$$\det(A) = \det\left([L_A]_\alpha^\alpha\right) = \det(L_A) \ .$$

The determinant of a linear transformation was independent of the choice of basis we are using to compute it. Let β be a basis with respect to which L_A has a diagonal matrix representation. Such a basis has to exist by part a). Then we have

$$\det(L_A) = \det\left([L_A]_\beta^\beta\right) = \det\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = (-1)\cdot 1 \cdot 2 = -2 \ .$$

With $\det(A) = -2$ we can now compute

$$\begin{aligned} \det(A^5) &= \det(A)^5 = (-2)^5 = -32 \ , \\ \det(A^t) &= \det(A) = -2 \ , \\ \det(3A) &= \det(3I_3)\cdot\det(A) = 3^3\cdot(-2) = -54 \ . \end{aligned}$$

Solution to 1d) Let $v \in \mathbb{R}^3$ be an eigenvector corresponding to the eigenvalue $\lambda \in \mathbb{R}$ for the matrix A. Then we have

$$(A + I_3)v = Av + v = \lambda\, v + v = (\lambda + 1)\, v \ .$$

Hence, v is also an eigenvector of $A + I_3$ corresponding to the eigenvalue $\lambda + 1$. Likewise, let $w \in \mathbb{R}^3$ be an eigenvector of $A + I_3$ corresponding to the eigenvalue $\mu \in \mathbb{R}$. Then

$$Aw = (A + I_3 - I_3)w = (A + I_3)w - w = \mu\, w - w \ .$$

Hence, w is also an eigenvector of A corresponding to the eigenvalue $\mu - 1$. Altogether we obtain a bijection between the set $\mathrm{EV}(A)$ of eigenvalues of A and the set $\mathrm{EV}(A + I_3)$ of eigenvalues of $A + I_3$ given by

$$\mathrm{EV}(A) \to \mathrm{EV}(A + I_3) \quad , \quad \lambda \mapsto \lambda + 1 \ .$$

In particular, the eigenvalues of $A + I_3$ are $\mu_1 = 0$, $\mu_2 = 2$ and $\mu_3 = 3$.

Note that it was important here that we looked at $A + I_3$ and not at $A + B$ for some other matrix $B \in M_{3\times 3}(\mathbb{R})$, since we used that all vectors are eigenvectors for I_3 corresponding to the eigenvalue 1.

Exercise 2 (Diagonalisation).
Let $T\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation given as follows

$$T(ax^2 + bx + c) = cx^2 + bx + a \ .$$

a) Determine the characteristic polynomial $p_T(t)$ of T and the eigenvalues of T.
b) Find a basis β for the vector space $P_2(\mathbb{R})$ such that $[T]_\beta^\beta$ is a diagonal matrix.

Solution to 2a) We know from the lecture that $\alpha = \{1, x, x^2\}$ is a basis for the 3-dimensional vector space $P_2(\mathbb{R})$ over the real numbers $\mathbb{R}$. The characteristic polynomial $p_T(t)$ is given by

$$p_T(t) = \det\left([T]_\alpha^\alpha - tI_3\right) \ .$$

Therefore we have to determine the matrix representation $[T]_\alpha^\alpha$ of T with respect to the basis α first. To do so we need to apply T to the basis elements. This yields

$$\begin{aligned} T(1) &= x^2 = 0 \cdot 1 + 0 \cdot x + 1 \cdot x^2 \ , \\ T(x) &= x = 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2 \ , \\ T(x^2) &= 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \ . \end{aligned}$$

The coefficients in these decompositions are the columns of the matrix $[T]_\alpha^\alpha$. Therefore we have

$$[T]_\alpha^\alpha = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and the characteristic polynomial is given by

$$p_T(t) = \det\left(\begin{pmatrix} -t & 0 & 1 \\ 0 & 1-t & 0 \\ 1 & 0 & -t \end{pmatrix}\right) = (-t)(-t)(1-t) - (1-t) = -(t+1)(1-t)^2$$

where we used cofactor expansion with respect to the first column to compute the determinant. The eigenvalues of T are the roots of the characteristic polynomial. Thus, there are two eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = -1$.

Solution to 2b) To find a basis β such that $[T]_\beta^\beta$ is a diagonal matrix we need to consider the two eigenspaces

$$\begin{aligned} E_1 &= \ker(T - I_{P_2(\mathbb{R})}) \ , \\ E_{-1} &= \ker(T + I_{P_2(\mathbb{R})}) \end{aligned}$$

and find bases for those. There are several ways to solve this problem. For example, to compute E_1 we could use the matrix representation of $T - I_{P_2(\mathbb{R})}$ with respect to the basis α. This is a matrix we essentially already computed above. It corresponds to $t = 1$. This gives

$$A = \left[T - I_{P_2(\mathbb{R})}\right]_\alpha^\alpha = [T]_\alpha^\alpha - I_3 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

Now we can compute the kernel $\ker(L_A)$ of the linear transformation $L_A \colon \mathbb{R}^3 \to \mathbb{R}^3$, which is the solution set of the following system of linear equations (over $\mathbb{R}$):

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a+c \\ 0 \\ a-c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

Thus, $\ker(L_A) = \left\{(a, b, c) \in \mathbb{R}^3 \mid a = c\right\}$, and the set $\{u_1, u_2\} \subset \ker(L_A)$ with $u_1 = (0, 1, 0)$, $u_2 = (1, 0, 1)$ is a basis for $\ker(L_A)$. How is this related to the vector space $E_1 = \ker(T - I_{P_2(\mathbb{R})})$? Note that by Thm. 2.0.14 we have

$$(T - I_{P_2(\mathbb{R})})(v) = 0 \quad \Leftrightarrow \quad \left[T - I_{P_2(\mathbb{R})}\right]_\alpha^\alpha [v]_\alpha = 0 \quad \Leftrightarrow \quad L_A([v]_\alpha) = 0$$

and hence

$$E_1 = \{v \in P_2(\mathbb{R}) \mid [v]_\alpha \in \ker(L_A)\} .$$

Moreover, it is not too difficult to check that if $\{u_1, u_2\}$ is a basis for $\ker(L_A)$, then the set $\{v_1, v_2\}$, where v_i is chosen such that $[v_i]_\alpha = u_i$, is a basis for E_1. However, $[v_1]_\alpha = u_1$ implies that $v_1 = x \in P_2(\mathbb{R})$ and $[v_2]_\alpha = u_2$ that $v_2 = 1 + x^2 \in P_2(\mathbb{R})$.

We could use the same approach to find a basis for E_{-1}, but we will choose a slightly different route for variety. By definition

$$E_{-1} = \ker(T + I_{P_2(\mathbb{R})})$$

Thus, the polynomial $p(x) = ax^2 + bx + c$ is in E_{-1} if and only if

$$(T + I_{P_2(\mathbb{R})})(p) = 0 \quad \Leftrightarrow \quad (c + a)x^2 + 2bx + (a + c) = 0 .$$

Comparing coefficients we obtain the conditions $a + c = 0$ and $b = 0$. Therefore

$$E_{-1} = \left\{ax^2 + 0x - a \mid a \in \mathbb{R}\right\} = \left\{a(x^2 - 1) \mid a \in \mathbb{R}\right\} \subset P_2(\mathbb{R}) .$$

This is a one-dimensional subspace with basis $\{x^2 - 1\}$.

Altogether, a basis β with the property that $[T]_\beta^\beta$ is a diagonal matrix is given by

$$\beta = \left\{x, 1 + x^2, x^2 - 1\right\} \ .$$

In fact, we have

$$[T]_\beta^\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ .$$

Exercise 3 (Generalised eigenspaces).
Let $L_A\colon \mathbb{C}^4 \to \mathbb{C}^4$ be the linear transformation given by the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Show that the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$ and find the eigenspaces E_2 and E_3 and the generalised eigenspaces $\bar{E}_2$ and $\bar{E}_3$.

Solution to 3) The characteristic polynomial of A is given by

$$p_A(t) = \det \begin{pmatrix} 2-t & 1 & 0 & 0 \\ 0 & 2-t & 0 & 0 \\ 0 & 0 & 3-t & 1 \\ 0 & 0 & 0 & 3-t \end{pmatrix} = (2-t)^2(3-t)^2 \ .$$

Since the eigenvalues of A are the roots of $p_A(t)$, they are indeed given by $\lambda_1 = 2$ and $\lambda_2 = 3$. The eigenspace E_2 of A is defined as $E_2 = \ker(L_B)$ with

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Given a vector $(x, y, z, w) \in \mathbb{C}^4$ it is an element in $\ker(L_B)$ if and only if it satisfies the equations:

$$\begin{aligned} y &= 0 \ , \\ z + w &= 0 \ , \\ w &= 0 \ . \end{aligned}$$

Therefore $E_2 = \ker(L_B) = \{a(1,0,0,0) \in \mathbb{C}^4 \mid a \in \mathbb{C}\} = \operatorname{span}\{e_1\}$ and a basis for E_2 is given by $\{e_1\}$. Likewise, $E_3 = \ker(L_C)$ with

$$C = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ .$$

A vector $(x, y, z, w) \in \mathbb{C}^4$ is in $\ker(L_C)$ if and only if

$$\begin{aligned} -x + y &= 0 \ , \\ -y &= 0 \ , \\ w &= 0 \ . \end{aligned}$$

The solution set is given by $E_3 = \ker(L_C) = \{a(0,0,1,0) \in \mathbb{C}^4 \mid a \in \mathbb{C}\}$ and a basis for E_3 is given by $\{e_3\}$.
Since the algebraic multiplicity of both eigenvalues is 2, we already know from the lecture that $\dim(\bar{E}_2) = \dim(\bar{E}_3) = 2$. The generalised eigenspace $\bar{E}_2$ was defined to be

$$\bar{E}_2 = \{v \in \mathbb{C}^4 \mid (A - 2I_4)^n v = 0 \text{ for some } n \in \mathbb{N}\} = \{v \in \mathbb{C}^4 \mid B^n v = 0 \text{ for some } n \in \mathbb{N}\} \ .$$

If we can find a vector $v \in \mathbb{C}^4$ such that $Bv = e_1$, then we have $v \in \bar{E}_2$, since

$$B^2 v = Be_1 = 0 \ .$$

Note that $v = e_2$ works and yields indeed $Be_2 = e_1$. Since $\dim(\bar{E}_2) = 2$ and $\{e_1, e_2\}$ is linearly independent, we must have $\bar{E}_2 = \text{span}\{e_1, e_2\}$.
Using the same technique we can find $\bar{E}_3$. First note that $Ce_4 = e_3$ and therefore $C^2 e_4 = 0$. This implies that $e_4 \in \bar{E}_3$. Since $\dim(\bar{E}_3) = 2$ and $\{e_3, e_4\}$ is linearly independent, we must have $\bar{E}_3 = \text{span}\{e_3, e_4\}$.

Exercise 4 (Diagonalisability).
Let $A \in M_{5\times 5}(\mathbb{C})$ and assume that

$$\lambda_1 = 2 \quad , \quad \lambda_2 = 4 \quad \text{and} \quad \lambda_3 = 6$$

are its only eigenvalues. Suppose that $\dim(E_{\lambda_1}) = 1$, $\dim(E_{\lambda_2}) = 2$ and $\dim(E_{\lambda_3}) = 1$. Is the matrix A diagonalisable? Explain your answer!

Solution to 4) The matrix A is not diagonalisable. To see why, we need to look at the algebraic multiplicities of the eigenvalues. Let m_i be the algebraic multiplicity of the eigenvalue λ_i. The sum of all these multiplicities is the dimension of the vector space the matrix acts on, so 5 in this case, i.e.

$$m_1 + m_2 + m_3 = 5 \ .$$

The sum of the dimensions of the eigenspaces is

$$\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) + \dim(E_{\lambda_3}) = 4$$

by assumption. If A were diagonalisable, then we would have $\dim(E_{\lambda_i}) = m_i$ by Thm. 2a) in Lec. 9. As we can see from the above equations, this can not be the case and we must have $\dim(E_{\lambda_i}) \neq m_i$ for at least one $i \in \{1, 2, 3\}$.

Exercise 5 (Division of polynomials).
In each of the following divide the polynomial p by the polynomial b, i.e. find a polynomial q such that $p = q \cdot b + r$ with b as given and a remainder polynomial r with $\deg(r) \leq \deg(b)$.

a) $p(x) = 12x^3 - 11x^2 + 9x + 18$ and $b(x) = 4x + 3$,

b) $p(x) = 3x^4 - 5x^2 + 3$ and $b(x) = x + 2$,

c) $p(x) = 4x^4 + 3x^3 + 2x + 1$ and $b(x) = x^2 + x + 2$.

Solution to 5) We just state the results here:

a) $p(x) = (3x^2 - 5x + 6)(4x + 3)$, i.e. $q(x) = 3x^2 - 5x + 6$ and $r(x) = 0$,

b) $p(x) = (3x^3 - 6x^2 + 7x - 14)(x + 2) + 31$, i.e. $q(x) = 3x^3 - 6x^2 + 7x - 14$ and $r(x) = 31$,

c) $p(x) = (4x^2 - x - 7)(x^2 + x + 2) + (11x + 15)$, i.e. $q(x) = (4x^2 - x - 7)$ and $r(x) = 11x + 15$.

Exercise 6 (Direct Sums).
Let V be a vector space and let $V_1, \dots, V_k$ be subspaces of V with the property that $V = V_1 \oplus \dots \oplus V_k$. Show that for any $v \in V$ there are vectors $v_i \in V_i$ for all $i \in \{1, \dots, k\}$ such that

$$v = v_1 + \dots + v_k \ .$$

Moreover, prove that this decomposition is unique, i.e. show that if $v_i' \in V_i$ for $i \in \{1, \dots, k\}$ is another set of vectors with $v = v_1' + \dots + v_k'$, then $v_i' = v_i$.

Solution to 6) It is part of the definition of the direct sum that

$$V = V_1 + \dots + V_k = \mathrm{span}(V_1 \cup \dots \cup V_k) \ .$$

This means that for each $v \in V$ we have vectors $v_i \in V_i$ with the property that $v = v_1 + \dots + v_k$. This solves the first part of the problem.
To see that there is only one such decomposition for every $v \in V$, we will first consider the special case $v = 0$. Let $v_i \in V_i$ be such that

$$0 = v_1 + \dots + v_k \ .$$

Fix $j \in \{1, \dots, k\}$ and bring v_j to the other side of the equation, i.e.

$$-v_j = v_1 + \dots + v_{j-1} + v_{j+1} + \dots + v_k \ .$$

The left hand side of the resulting equation is a vector in V_j, whereas the right hand side is in the vector space $V_1 + \dots + V_{j-1} + V_{j+1} + \dots + V_k$. Therefore both sides are in the intersection of the two vector spaces, i.e. in

$$V_j \cap (V_1 + \dots + V_{j-1} + V_{j+1} + \dots + V_k) \ .$$

But by the definition of the direct sum each of these intersections is the zero vector space. This means that all v_j have to vanish and the decomposition of $0 \in V$ is unique.
Now let $v \in V$ be an arbitrary vector and suppose that $v = v_1 + \dots + v_k = v_1' + \dots + v_k'$ with $v_i, v_i' \in V_i$ are two decompositions of V. Then the sum

$$(v_1 - v_1') + \dots + (v_k - v_k') = v - v = 0$$

is a decomposition of the zero vector (note that $v_i - v_i' \in V_i$, since V_i is a vector space). By our previous argument this decomposition is unique, which means that all the summands have to vanish, i.e. $v_i = v_i'$ for all $i \in \{1, \dots, k\}$. It follows that the two decompositions of v have to agree.

Exercise 7 (Jordan normal form).
Let $L_A\colon \mathbb{C}^4 \to \mathbb{C}^4$ be the linear transformation given by the matrix

$$A = \begin{pmatrix} 7 & 1 & -8 & -1 \\ 0 & 3 & 0 & 0 \\ 4 & 2 & -5 & -1 \\ 0 & -4 & 0 & -1 \end{pmatrix}$$

a) Determine the characteristic polynomial $p_A(t)$, the eigenvalues of L_A and their multiplicities.
b) Find a basis for each of the eigenspaces E_λ.
c) Determine a Jordan basis for A, i.e. find a basis β for $\mathbb{C}^4$ with the property that $[L_A]_\beta^\beta$ is in Jordan normal form.

Solution to 7a) The characteristic polynomial of A is given by

$$\det \begin{pmatrix} 7-t & 1 & -8 & -1 \\ 0 & 3-t & 0 & 0 \\ 4 & 2 & -5-t & -1 \\ 0 & -4 & 0 & -1-t \end{pmatrix} = (3-t)\det \begin{pmatrix} 7-t & -8 & -1 \\ 4 & -5-t & -1 \\ 0 & 0 & -1-t \end{pmatrix}$$
$$=(t-3)(t+1)\det \begin{pmatrix} 7-t & -8 \\ 4 & -5-t \end{pmatrix} = (t-3)(t+1)\left[(7-t)(-5-t)+32\right]$$
$$=(t-3)(t+1)(t^2-2t-3) = (t-3)^2(t+1)^2 \ ,$$

where we used a cofactor expansion along the second row for the first equality and the cofactor expansion along the third row for the second equality. The roots of $p_A(t)$ are $\lambda_1 = 3$ and $\lambda_2 = -1$. These agree with the eigenvalues of L_A. The multiplicity of λ_1 is 2 and the multiplicity of λ_2 is 2 as well.

Solution to 7b) To determine the eigenspace E_3 of $\lambda_1 = 3$ have to determine the kernel of L_B with

$$B = \begin{pmatrix} 4 & 1 & -8 & -1 \\ 0 & 0 & 0 & 0 \\ 4 & 2 & -8 & -1 \\ 0 & -4 & 0 & -4 \end{pmatrix}$$

Using elementary row operations we can transform this matrix into

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Therefore the kernel is

$$E_3 = \left\{(2a, 0, a, 0) \in \mathbb{C}^4 \mid a \in \mathbb{C}\right\} = \{a(2, 0, 1, 0) \mid a \in \mathbb{C}\} .$$

Any set containing just one non-zero vector from E_3 forms a basis for E_3. For example, we can choose $\beta_1 = \{v_1\}$ with $v_1 = (2, 0, 1, 0)$.

The eigenspace E_{-1} can be computed in a similar way. It agrees with the kernel of L_C with

$$C = \begin{pmatrix} 8 & 1 & -8 & -1 \\ 0 & 4 & 0 & 0 \\ 4 & 2 & -4 & -1 \\ 0 & -4 & 0 & 0 \end{pmatrix} .$$

After some elementary row operations this matrix turns into

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which implies that the kernel is

$$E_{-1} = \left\{a(1, 0, 1, 0) \in \mathbb{C}^4 \mid a \in \mathbb{C}\right\}$$

and $\beta_2 = \{v_3\}$ with $v_3 = (1, 0, 1, 0)$ defines a basis. It will become clear in part c), why we call this vector v_3 instead of v_2.

Solution to 7c) The multiplicity of both eigenvalues is 2, whereas the dimension of both eigenspaces is 1. By a theorem in the lecture the dimension of the generalised eigenspaces is equal to the multiplicity of the corresponding eigenvalue. Therefore the generalised eigenspaces $\bar{E}_3$ and $\bar{E}_{-1}$ are both 2-dimensional.

To extend the basis β_1 from b) to a basis of the generalised eigenspace $\bar{E}_3$ such that L_A is in Jordan normal form we have to find a vector $v_2 \in \mathbb{C}^4$ which solves the equation

$$(L_A - 3I_4)(v_2) = L_B(v_2) = v_1$$

This means we have to solve the following system of linear equations given in augmented matrix form:

$$\left(\begin{array}{cccc|c} 4 & 1 & -8 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & -8 & -1 & 1 \\ 0 & -4 & 0 & -4 & 0 \end{array}\right) \quad \rightsquigarrow \quad \left(\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

We arrived at the right hand side via elementary row operations. From this we see that the vector $v_2 = (3, -1, 1, 1)$ is a solution of this system.
We also have to extend $\beta_2 = \{v_3\}$ to a basis for $\bar{E}_{-1}$. To achieve this we have to find a vector $v_4 \in \mathbb{C}^4$ that solves the equation

$$(L_A + 3I_4)(v_4) = L_C(v_4) = v_3 \ .$$

The corresponding augmented matrix in this case is

$$\left(\begin{array}{cccc|c} 8 & 1 & -8 & -1 & 1 \\ 0 & 4 & 0 & 0 & 0 \\ 4 & 2 & -4 & -1 & 1 \\ 0 & -4 & 0 & 0 & 0 \end{array}\right) \quad \rightsquigarrow \quad \left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

In particular, $v_4 = (0, 0, 0, -1)$ is a solution of this system.
The set $\beta = \{v_1, v_2, v_3, v_4\}$ is a basis for $\mathbb{C}^4$. The corresponding matrix representation is

$$[L_A]_\beta^\beta = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and it is in Jordan normal form.

Exercise 8 (Nilpotent linear transformations).

In this exercise we will look at some other properties of nilpotent linear transformations.

a) Find an example of a vector space V and two nilpotent linear transformations $S\colon V \to V$ and $T\colon V \to V$ with the property that $S \circ T$ is no longer nilpotent.

b) Let V be a finite-dimensional vector space and let $S\colon V \to V$ and $T\colon V \to V$ be nilpotent linear transformations with the property $T \circ S = S \circ T$. Prove that $S \circ T$ is in this case also nilpotent.

c) Is there a finite-dimensional vector space V together with two nilpotent linear transformations $S\colon V \to V$ and $T\colon V \to V$ such that $S \circ T$ is invertible?

Solution to 8a) Let $V = \mathbb{C}^2$ and consider $T = L_A\colon \mathbb{C}^2 \to \mathbb{C}^2$ with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Likewise, let $S = L_B\colon \mathbb{C}^2 \to \mathbb{C}^2$ be the linear transformation with

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Since $A^2 = 0 = B^2$ both transformations are nilpotent. However, $S \circ T = L_{BA}$ is given by left multiplication with

$$BA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and we have $(BA)^n = BA \neq 0$ for all $n \in \mathbb{N}$. Therefore $S \circ T$ is no longer nilpotent.

Solution to 8b) If $T \circ S = S \circ T$, then we have

$$(ST)^2 = (ST)(ST) = S(TS)T = S(ST)T = S^2T^2$$

and by induction we have $(ST)^n = S^nT^n$. Indeed, if we assume that the statement is true for $n-1$, then

$$(ST)^n = (ST)^{n-1}ST = S^{n-1}T^{n-1}ST = S^nT^n \ ,$$

where we used that we can move S past each T in the power T^{n-1} to arrive at the last equation. Since S is nilpotent there is an $n \in \mathbb{N}$ with $S^n = 0$. But then

$$(ST)^n = S^n \circ T^n = 0$$

and ST is nilpotent. Note that we did not use that T is nilpotent here.

Solution to 8c) No. This can not happen, since ST is invertible if and only if $\det(ST) \neq 0$. But the determinant is multiplicative. Therefore $\det(ST) = \det(S) \cdot \det(T)$. Neither S nor T is invertible, since they are nilpotent linear transformations. Hence, $\det(S) = \det(T) = 0$, which implies $\det(ST) = 0$.

Exercise 9 (Nilpotent linear transformations).
Let $A \in M_{8\times 8}(\mathbb{C})$ be a matrix with the following properties:

- $L_A \colon \mathbb{C}^8 \to \mathbb{C}^8$ is nilpotent,
- $\operatorname{rank}(A) = 5$,
- $\operatorname{rank}(A^2) = 2$.

a) Show that the only possible eigenvalue of A is 0.

b) List all the possible Jordan normal forms (up to permutation of the Jordan blocks) you could get for $[L_A]_\beta^\beta$.

Solution to 9a) Since L_A is nilpotent, there is an $n \in \mathbb{N}$ with the property that $L_A^n = 0$. If $v \in \mathbb{C}^8$ is an eigenvector with corresponding eigenvalue λ, then

$$0 = L_A^n(v) = \lambda^n v$$

Since $v \neq 0$, this implies $\lambda^n = 0$, which gives $\lambda = 0$. Thus, we have shown that any eigenvalue of A has to be zero. To see that A actually has eigenvalues, we could argue as follows: The characteristic polynomial $p_A(t)$ is a polynomial of degree n with coefficients in the complex numbers. As such it has at least one root.
There is another way of seeing that there exists at least one eigenvector with eigenvalue 0: Such an eigenvector is a non-zero element in $\ker(L_A)$. Suppose for the sake of contradiction that $\ker(L_A) = \{0\}$. But then L_A is an isomorphism. Since compositions of isomorphisms are again isomorphisms, we also have that L_A^m is an isomorphism for each $m \in \mathbb{N}$. But this can not be, since there is an $n \in \mathbb{N}$ with $L_A^n = 0$.

Solution to 9b) By the rank-nullity theorem we have

$$\dim(\ker(L_A)) = \dim(\mathbb{C}^8) - \operatorname{rank}(A) = 8 - 5 = 3\ .$$

This means that we have three cycles of generalised eigenvectors in the Jordan basis of the nilpotent linear transformation L_A. Denote the sizes of the corresponding Jordan blocks by n_1, n_2 and n_3. We sort them by demanding $n_1 \leq n_2 \leq n_3$. If we sum up the sizes of the blocks, we end up with the dimension of the vector space $\mathbb{C}^8$, i.e.

$$n_1 + n_2 + n_3 = 8\ .$$

Let $J_k(0) \in M_{k\times k}(\mathbb{C})$ be a Jordan block of size k with zeroes along the diagonal. Observe that

$$\operatorname{rank}(J_k(0)) = k - 1\ ,$$

since it has $k-1$ linearly independent columns. If we take the square of the block, then it looks like this

$$J_k(0)^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} ,$$

that is, the line of ones above the diagonal moves to the second line above the diagonal and we have two columns of zeroes on the left and two rows of zeroes at the bottom. In particular, if $k \geq 2$, then

$$\operatorname{rank}(J_k(0)^2) = k - 2 ,$$

and if $k < 2$, then $\operatorname{rank}(J_k(0)^2) = 0$. Now suppose that β is a Jordan basis for A, such that

$$[L_A]_\beta^\beta = \begin{pmatrix} J_{n_1}(0) & 0 & 0 \\ 0 & J_{n_2}(0) & 0 \\ 0 & 0 & J_{n_3}(0) \end{pmatrix}$$

Then we have

$$[L_{A^2}]_\beta^\beta = \begin{pmatrix} J_{n_1}(0)^2 & 0 & 0 \\ 0 & J_{n_2}(0)^2 & 0 \\ 0 & 0 & J_{n_3}(0)^2 \end{pmatrix}$$

and therefore

$$2 = \operatorname{rank}(A^2) = \operatorname{rank}\left([L_{A^2}]_\beta^\beta\right) = \operatorname{rank}(J_{n_1}(0)^2) + \operatorname{rank}(J_{n_2}(0)^2) + \operatorname{rank}(J_{n_3}(0)^2) . \quad (1)$$

Suppose first that $n_1 < 2$. Then we must have $n_1 = 1$, since the size can not be zero. From this we deduce $n_2 + n_3 = 7$. Note that this means we can not have $n_2 = n_3 = 1$ as well. By our observations above, we must have $\operatorname{rank}(J_{n_1}(0)^2) = 0$ in this case. If we assume that $n_2 = 1$, then $n_3 = 6$ and the right hand side of (1) gives $6 - 2 = 4$, which is a contradiction. Thus, we can assume $n_2 \geq 2$. But then the right hand side of (1) gives $n_2 - 2 + n_3 - 2 = 7 - 4 = 3$, which is still a contradiction.
Altogether we see that our initial assumption must have been wrong and we can assume $n_1 \geq 2$, which implies $n_2 \geq 2$ and $n_3 \geq 2$ as well. In this case the right hand side of (1) gives

$$n_1 - 2 + n_2 - 2 + n_3 - 2 = 8 - 6 = 2 .$$

Since there is no contradiction any more, this must characterise the possible Jordan normal forms. So, we are looking for natural numbers $2 \leq n_1 \leq n_2 \leq n_3$ with $n_1 + n_2 + n_3 = 8$. The only possibilities are $(n_1, n_2, n_3) = (2, 2, 4)$ and $(n_1, n_2, n_3) = (2, 3, 3)$. The corresponding Jordan normal forms are

$$[L_{A^2}]_\beta^\beta = \begin{pmatrix} J_2(0) & 0 & 0 \\ 0 & J_2(0) & 0 \\ 0 & 0 & J_4(0) \end{pmatrix} \quad , \quad [L_{A^2}]_\beta^\beta = \begin{pmatrix} J_2(0) & 0 & 0 \\ 0 & J_3(0) & 0 \\ 0 & 0 & J_3(0) \end{pmatrix} .$$