Solutions to the problem sheet for

MA2008 – Linear Algebra II

Sheet 7 Autumn Semester 2019

Exercise 1 (Eigenvalues and eigenvectors).
Find two different matrices $A \in M_{2\times 2}(\mathbb{R})$ that have the vector $v = (1,1) \in \mathbb{R}^2$ as an eigenvector with corresponding eigenvalue 2.

Solution to 1. Let $A \in M_{2\times 2}(\mathbb{R})$, then A has the form

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} .$$

with $a_{ij} \in \mathbb{R}$. We want v to be an eigenvector of A with corresponding eigenvalue 2. Hence, we must have

$$Av = 2v \quad \Leftrightarrow \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

This gives the following two equations:

$$a_{11} + a_{12} = 2 ,$$
$$a_{21} + a_{22} = 2 .$$

Any matrix with entries a_{ij} that satisfy the above equations, will have the vector $v = (1,1)$ as an eigenvector with corresponding eigenvalue 2. We can for example take

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad , \quad A_2 = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} .$$

Exercise 2 (Jordan normal form).
Let $A \in M_{3\times3}(\mathbb{C})$ be the following matrix:

$$A = \begin{pmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 5 & 0 & 0 \end{pmatrix}$$

a) Determine the characteristic polynomial $p_A(t)$ and the eigenvalues of A.

b) For each eigenvalue λ of A find a basis for the corresponding eigenspace E_λ.

c) Find a Jordan basis for A and determine its Jordan normal form.

Solution to 2a) By definition the characteristic polynomial of A is given by

$$\begin{aligned} p_A(t) &= \det \begin{pmatrix} -2-t & 2 & 1 \\ -7 & 4-t & 2 \\ 5 & 0 & -t \end{pmatrix} = 5\det\begin{pmatrix} 2 & 1 \\ 4-t & 2 \end{pmatrix} - t\det\begin{pmatrix} -2-t & 2 \\ -7 & 4-t \end{pmatrix} \\ &= 5t + t(-t^2+2t-6) = -t(t^2-2t+1) = -t(t-1)^2 \ . \end{aligned}$$

Therefore the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 1$.

Solution to 2b) To determine the eigenspace corresponding to the eigenvalue 0 we have to solve

$$\begin{pmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 5 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

The solution set of this system of linear equations is given by

$$\begin{aligned} E_0 &= \{(x,y,z) \in \mathbb{C}^3 \mid x = 0,\ 2y + z = 0\} \\ &= \{(0,a,-2a) \in \mathbb{C}^3 \mid a \in \mathbb{C}\} \\ &= \operatorname{span}\{(0,1,-2)\} \end{aligned}$$

and a basis for this vector space is for example given by $\{v_1\}$ with $v_1 = (0,1,-2)$.
To determine the eigenspace corresponding to the eigenvalue 1 we have to solve

$$\begin{pmatrix} -3 & 2 & 1 \\ -7 & 3 & 2 \\ 5 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

Using elementary row operations we obtain the following augmented matrix:

$$\left(\begin{array}{ccc|c} -3 & 2 & 1 & 0 \\ -7 & 3 & 2 & 0 \\ 5 & 0 & -1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} -3 & 2 & 1 & 0 \\ 3 & 3 & 0 & 0 \\ 5 & 0 & -1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} -3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 5 & 0 & -1 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & -5 & -1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set of this system of linear equations is therefore given by

$$\begin{aligned} E_1 &= \{(x,y,z) \in \mathbb{C}^3 \mid x+y=0,\ 5y+z=0\} \\ &= \{(-a,a,-5a) \in \mathbb{C}^3 \mid a \in \mathbb{C}\} \\ &= \operatorname{span}\{(-1,1,-5)\} \end{aligned}$$

and a basis for this vector space is for example given by $\{v_2\}$ with $v_2 = (-1,1,-5)$.

Solution to 2c) Note that $\dim(E_0) = 1$ agrees with the algebraic multiplicity of $\lambda_1 = 0$, which is 1. However, the algebraic multiplicity of $\lambda_2 = 1$ is 2, whereas $\dim(E_1) = 1$. Hence, A is not diagonalisable and the Jordan normal form has to be

$$[L_A]_\beta^\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} .$$

To complete the set $\{v_1, v_2\}$ to a Jordan basis, we need to find a vector $v_3 \in \mathbb{C}^3$ with the property that

$$(A - \lambda_2 I_3)v_3 = v_2 ,$$

i.e. we need to solve the system of linear equations

$$\begin{pmatrix} -3 & 2 & 1 \\ -7 & 3 & 2 \\ 5 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -5 \end{pmatrix} .$$

We can achieve this by using the same elementary row operations as above:

$$\left(\begin{array}{ccc|c} -3 & 2 & 1 & -1 \\ -7 & 3 & 2 & 1 \\ 5 & 0 & -1 & -5 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} -3 & 2 & 1 & -1 \\ 3 & 3 & 0 & -9 \\ 5 & 0 & -1 & -5 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} -3 & 2 & 1 & -1 \\ 1 & 1 & 0 & -3 \\ 5 & 0 & -1 & -5 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & -3 \\ 0 & 5 & 1 & -10 \\ 0 & -5 & -1 & 10 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & -3 \\ 0 & 5 & 1 & -10 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since we are only interested in a single solution, we can stop here and consider the resulting equations

$$\begin{aligned} x + y &= -3 \ , \\ 5y + z &= -10 \ . \end{aligned}$$

This is solved by the vector $v_3 = (-1, -2, 0)$. Hence, $\beta = \{v_1, v_2, v_3\}$ is a Jordan basis for A.

Exercise 3 (Inner products).
Let $V = \mathbb{R}^2$ and define $\langle\cdot\,,\cdot\rangle\colon V\times V\to\mathbb{R}$ by

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \frac{1}{2}\left(3x_1x_2 + y_1x_2 + x_1y_2 + 3y_1y_2\right) .$$

Show that this defines an inner product on V.

Solution to 3. First note that

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \frac{1}{2}\left(3x_1x_2 + y_1x_2 + x_1y_2 + 3y_1y_2\right) ,$$
$$\left\langle \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \right\rangle = \frac{1}{2}\left(3x_2x_1 + y_2x_1 + x_2y_1 + 3y_2y_1\right)$$

and since both right hand sides agree, the expression is symmetric. Now let $(x_2', y_2') \in \mathbb{R}^2$ and let $a \in \mathbb{R}$. Then we have

$$\begin{aligned}
&\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + a\begin{pmatrix} x_2' \\ y_2' \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 + ax_2' \\ y_2 + ay_2' \end{pmatrix} \right\rangle \\
=&\frac{1}{2}\left(3x_1(x_2+ax_2') + y_1(x_2+ax_2') + x_1(y_2+ay_2') + 3y_1(y_2+ay_2')\right) \\
=&\frac{1}{2}\left(3x_1x_2 + a3x_1x_2' + y_1x_2 + ay_1x_2' + x_1y_2 + ax_1y_2' + 3y_1y_2 + a3y_1y_2'\right) \\
=&\frac{1}{2}\left(3x_1x_2 + y_1x_2 + x_1y_2 + 3y_1y_2\right) + a\frac{1}{2}\left(3x_1x_2' + y_1x_2' + x_1y_2' + 3y_1y_2'\right) \\
=&\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle + a\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2' \\ y_2' \end{pmatrix} \right\rangle
\end{aligned}$$

Therefore our definition produces a symmetric bilinear form. To see that it is positive definite we have to compute it in the case when $(x_1, y_1) = (x_2, y_2) = (x, y)$. This gives

$$\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \frac{1}{2}\left(3x^2 + 2xy + 3y^2\right) = \frac{1}{2}\left(2x^2 + (x+y)^2 + 2y^2\right) .$$

All summands on the right hand side are squares. Therefore their sum is greater or equal to zero. Moreover, if the right hand side vanishes, then all summands have to be zero by themselves. This implies $x = y = 0$, which shows that our symmetric bilinear form is positive definite.

Exercise 4 (The trace of a matrix).
For a matrix $A \in M_{n\times n}(\mathbb{R})$ with entries $a_{ij} \in \mathbb{R}$ the trace of A is given by

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii} \ .$$

a) Show that $\operatorname{tr}\colon M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ is a linear transformation.
b) Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for any $A, B \in M_{n\times n}(\mathbb{R})$.
c) Show that $\operatorname{tr}(A^t A) \geq 0$ and that $\operatorname{tr}(A^t A) = 0$ if and only if $A = 0$.

Solution to 4a) Let $A, B \in M_{n\times n}(\mathbb{R})$. Denote the entries of A by a_{ij} and the ones of B by b_{ij}. Let $c \in \mathbb{R}$. To see that the trace is linear it suffices to show that

$$\operatorname{tr}(A + cB) = \operatorname{tr}(A) + c\operatorname{tr}(B) \ .$$

Using the definition of the trace we now get

$$\operatorname{tr}(A + cB) = \sum_{i=1}^{n}(a_{ii} + c\, b_{ii}) = \sum_{i=1}^{n} a_{ii} + c\sum_{i=1}^{n} b_{ii} = \operatorname{tr}(A) + c\operatorname{tr}(B) \ .$$

This shows the linearity of the trace.

Solution to 4b) Let $A, B \in M_{n\times n}(\mathbb{R})$ and denote the entries again by a_{ij} and b_{ij} respectively. Let $C = AB$ and denote its entries by c_{ij}. Then we have

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \ .$$

Likewise let $D = BA$ and denote the entries of this matrix by d_{ij}. Here we get

$$d_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj} \ .$$

If we take the trace of AB we obtain

$$\begin{aligned}\operatorname{tr}(AB) = \operatorname{tr}(C) &= \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik} b_{ki} = \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ki} a_{ik} \\ &= \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ik} a_{ki} = \sum_{i=1}^{n} d_{ii} = \operatorname{tr}(D) = \operatorname{tr}(BA) \ ,\end{aligned}$$

where we interchanged the summation indices i and k from the first line to the second.

Solution to 4c) Let $A \in M_{n\times n}(\mathbb{R})$ and denote the entries of A by a_{ij}. Let $C = A^t A$ and denote its entries by c_{ij}. Then we have

$$c_{ij} = \sum_{k=1}^{n} a_{ki} a_{kj} \ .$$

Taking the trace of $A^t A$ we therefore get

$$\mathrm{tr}(A^t A) = \mathrm{tr}(C) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki}^2 \ .$$

This is a sum of squares of real numbers and therefore always greater or equal to 0. If $\mathrm{tr}(A^t A) = 0$, then we must have $a_{ki} = 0$ for all $i, k \in \{1, \ldots, n\}$. Hence, $A = 0$ in this case.

Exercise 5 (Duals of linear transformations).

Let V and W be finite-dimensional vector spaces over $\mathbb{R}$. Let $T\colon V \to W$ be a linear transformation and let

$$T^*\colon W^* \to V^* \quad , \quad \varphi \mapsto \varphi \circ T \ .$$

a) Show that $T^*(\varphi)$ is indeed an element of V^* for all $\varphi \in W^*$.

b) Show that T^* is a linear transformation.

c) Suppose that $\dim(V) = n$ and $\dim(W) = m$ respectively. Let $\beta = \{v_1, \ldots, v_n\}$ be a basis for V and let $\gamma = \{w_1, \ldots, w_m\}$ be a basis for W. Denote the dual bases by β^* and γ^* respectively. Prove that

$$[T^*]_{\gamma^*}^{\beta^*} = \left([T]_\beta^\gamma\right)^t \ .$$

Solution to 5a) To see that $T^*(\varphi) \in V^*$ we need to show that it is a linear transformation. However, $T\colon V \to W$ is a linear transformation and $\varphi\colon W \to \mathbb{R}$ is an element of W^* and therefore in particular also a linear transformation. But then $T^*(\varphi) = \varphi \circ T$ is a linear transformation, since it is the composition of two linear transformations.

Solution to 5b) Let $\varphi_1, \varphi_2 \in W^*$ and let $c \in \mathbb{R}$. To see that T^* is a linear transformation we need to check that $T^*(\varphi_1 + c\,\varphi_2) = T^*(\varphi_1) + c\,T^*(\varphi_2)$. Let $v \in V$. Then we have

$$\begin{aligned} T^*(\varphi_1 + c\,\varphi_2)(v) &= (\varphi_1 + c\,\varphi_2)(T(v)) = \varphi_1(T(v)) + c\,\varphi_2(T(v)) \\ &= (\varphi_1 \circ T)(v) + c\,(\varphi_2 \circ T)(v) = (T^*(\varphi_1) + c\,T^*(\varphi_2))(v) \ . \end{aligned}$$

Since this is true for any vector $v \in V$, we must have $T^*(\varphi_1 + c\,\varphi_2) = T^*(\varphi_1) + c\,T^*(\varphi_2)$.

Solution to 5c) To determine the ith column of the matrix representation of T^* with respect to the bases γ^* and β^*, we need to write $T^*(w_i^*)$ as a linear combination of the basis elements $v_1^*, \ldots, v_n^*$. We make the following ansatz:

$$T^*(w_i^*) = \sum_{\ell=1}^{n} \lambda_\ell^i \, v_\ell^* \ . \tag{1}$$

We now have to determine the coefficients $\lambda_\ell^i \in \mathbb{R}$. Let $A = [T]_\beta^\gamma$ and denote the entries of A by a_{ij}. By the definition of the matrix representation we have

$$T(v_j) = \sum_{k=1}^{m} a_{kj} w_k \ .$$

If we evaluate the right hand side of (1) on the vector $v_j \in V$ we obtain

$$\sum_{\ell=1}^{n} \lambda_\ell^i \, v_\ell^*(v_j) = \lambda_j^i$$

because $v_\ell^*(v_j)$ vanishes for $\ell \neq j$ and $v_j^*(v_j) = 1$. If we evaluate the left hand side of (1) on v_j we get

$$T^*(w_i^*)(v_j) = w_i^*(T(v_j)) = w_i^* \left(\sum_{k=1}^{m} a_{kj} w_k \right) = \sum_{k=1}^{m} a_{kj} w_i^*(w_k) = a_{ij} \ ,$$

since again $w_i^*(w_k)$ vanishes for $i \neq k$ and $w_i^*(w_i) = 1$. Therefore $\lambda_j^i = a_{ij}$. The ith column of $[T^*]_{\gamma^*}^{\beta^*}$ is given by $(\lambda_1^i, \ldots, \lambda_n^i) = (a_{i1}, \ldots, a_{in})$. However, $(a_{i1}, \ldots, a_{in})$ is also the ith row of A. Therefore

$$[T^*]_{\gamma^*}^{\beta^*} = \left([T]_\beta^\gamma \right)^t \ .$$

Exercise 6 (Gram–Schmidt orthonormalisation process).
Consider the vector space $\mathbb{R}^3$ equipped with the dot product as inner product. Use the Gram–Schmidt orthonormalisation process to transform $\{u_1, u_2, u_3\} \subset \mathbb{R}^3$ with

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad , \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad , \quad u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ .$$

into an orthonormal basis.

Solution to 6) The first step of the Gram–Schmidt process consists of rescaling the vector u_1 appropriately. We have

$$\|u_1\| = \sqrt{\langle u_1, u_1 \rangle} = \sqrt{3}$$

and therefore

$$f_1 = \frac{1}{\sqrt{3}} u_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \ .$$

In the next step we have to subtract the orthogonal projection of u_2 onto f_1 from u_2, i.e. we need to compute

$$\begin{aligned} f_2' &= u_2 - \langle f_1, u_2 \rangle \, f_1 \\ &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \ . \end{aligned}$$

We need to rescale this vector appropriately. To do this we have to compute its length. Its square is given by

$$\langle f_2', f_2' \rangle = \frac{1}{9}(1 + 1 + 4) = \frac{6}{9} = \frac{2}{3} \ .$$

Therefore we have

$$f_2 = \frac{1}{\|f_2'\|} f_2' = \sqrt{\frac{3}{2}} \cdot \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \ .$$

Finally, we have to repeat this procedure with u_3 and consider

$$\begin{aligned} f_3' &= u_3 - \langle f_1, u_3 \rangle \, f_1 - \langle f_2, u_3 \rangle \, f_2 \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \ , \end{aligned}$$

which leads to

$$f_2 = \frac{1}{\|f_2'\|} f_2' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

The set $\beta = \{f_1, f_2, f_3\}$ is the orthonormal basis we were looking for.

Exercise 7 (Linear forms and kernels).
Let V be a vector space over the field F. Let $\alpha, \beta \in V^*$ be linear forms on V with the property that $\alpha(v) = 0$ if and only if $\beta(v) = 0$. Show that in this situation we have $\alpha = \lambda\,\beta$ for some $\lambda \in F$.

Solution to 7) If $\alpha = 0$ then we must have $\beta = 0$ as well, and the statement is trivially true. Therefore we can assume that $\alpha\colon V \to F$ is not the zero map. This means that there is at least one vector $v \in V$ with the property that $\alpha(v) \neq 0$. Let

$$w = \frac{1}{\alpha(v)}\, v \ .$$

This vector satisfies $\alpha(w) = 1$. Let $\lambda \in F$. Then we have

$$\alpha(\lambda\, w) = \lambda\, \alpha(w) = \lambda \ .$$

Since $\lambda \in F$ was arbitrary, we see that α is surjective. Now define a linear transformation $\psi\colon F \to F$ as follows: For $\lambda \in F$ let

$$\psi(\lambda) = \beta(v_\lambda)$$

where $v_\lambda \in V$ is any vector with $\alpha(v_\lambda) = \lambda$. We first need to check that ψ does not depend on the choice of the vector v_λ. Suppose that $v'_\lambda \in V$ satisfies $\alpha(v'_\lambda) = \lambda$. Then we have

$$\alpha(v_\lambda - v'_\lambda) = \alpha(v_\lambda) - \alpha(v'_\lambda) = \lambda - \lambda = 0 \ .$$

By our assumptions on α and β this implies that

$$0 = \beta(v_\lambda - v'_\lambda) = \beta(v_\lambda) - \beta(v'_\lambda) \quad \Leftrightarrow \quad \beta(v_\lambda) = \beta(v'_\lambda) \ .$$

So even though v_λ and v'_λ might be distinct vectors, $\beta(v_\lambda) = \beta(v'_\lambda)$ and $\psi(\lambda)$ does not depend on the choice of v_λ.
We also need to see that ψ is a linear transformation. Let $\lambda, \mu, c \in F$. Pick v_λ such that $\alpha(v_\lambda) = \lambda$ and v_μ such that $\alpha(v_\mu) = \mu$. Now note that

$$\alpha(v_\lambda + c\, v_\mu) = \lambda + c\, \mu$$

and therefore we can pick $v_{\lambda + c\,\mu} = v_\lambda + c\, v_\mu$. But then we have

$$\psi(\lambda + c\,\mu) = v_{\lambda + c\,\mu} = v_\lambda + c\, v_\mu = \psi(\lambda) + c\, \psi(\mu)$$

and ψ is indeed a linear transformation.

Now observe that linearity of $\psi\colon F \to F$ implies that

$$\psi(\lambda) = \psi(1)\,\lambda\ .$$

Therefore ψ is given by the multiplication by the scalar $\lambda_\psi = \psi(1) \in F$.
Now let $v \in V$ be an arbitrary vector. We want to compute $\psi(\alpha(v))$ for $\lambda = \alpha(v)$. To do so we must find a vector v_λ with the property that $\alpha(v_\lambda) = \alpha(v)$. But this is satisfied by $v_\lambda = v$. Therefore we have

$$\lambda_\psi\,\alpha(v) = \psi(\alpha(v)) = \beta(v_{\alpha(v)}) = \beta(v)\ .$$

Since this holds for any vector $v \in V$, we must have $\lambda_\psi \alpha = \beta$. Note that in this case $\lambda_\psi \neq 0$, because we assumed that $\alpha \neq 0$, which implies that $\beta \neq 0$. Thus, $\lambda_\psi \neq 0$ and we also have

$$\beta = \frac{1}{\lambda_\psi}\,\alpha\ .$$