Problem sheet for the lecture

MA3008 – Algebraic Topology

Sheet 1 Spring Semester 2020

Exam Style Questions

Exercise 1 (Open sets and open balls).
Let (X, d) be a metric space. Show that a subset $U \subset X$ is open with respect to the metric d if and only if it can be written as a union of open balls.

Exercise 2 (Open subsets of $\mathbb{R}^2$).
Consider the plane $\mathbb{R}^2$ equipped with its standard metric d given by

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Determine whether the following subsets of $\mathbb{R}^2$ are open with respect to the metric d. Justify your answers.

a) $A = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\}$,
b) $B = \{(x, y) \in \mathbb{R}^2 \mid x = 0\}$,
c) $C = \{(x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y < 5\}$.

Exercise 3 (A strange metric on an arbitrary set).
Let X be a set and define

$$d \colon X \times X \to \mathbb{R} \quad , \quad (x, y) \mapsto \begin{cases} 0 & \text{if } x = y \ , \\ 1 & \text{else } . \end{cases}$$

Show that (X, d) is a metric space.

Exercise 4 (A function on $\mathbb{R}^2$ that is nearly a metric).
Consider the following function

$$d \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \quad , \quad ((x_1, y_1), (x_2, y_2)) \mapsto |x_1 - x_2| \ ,$$

This function is not a metric on $\mathbb{R}^2$. Which of the properties of a metric does d satisfy?

Intermediate Level Questions

Exercise 5 (ε-δ-Definition of continuity).
Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{else} \end{cases} .$$

a) Show that f is continuous at $x = 0$.

b) Show that f is not continuous at any other point $x \neq 0$.

Exercise 6 (Open sets in metric spaces).
In the following list (X, d) is a metric space and $U \subset X$ is a subset. Decide whether U is open with respect to the metric d.

a) $X = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\} \subset \mathbb{R}^2$ equipped with the metric it inherits from $\mathbb{R}^2$, i.e.

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

and $U = \{(x, y) \in \mathbb{R}^2 \mid x \cdot y < 1 \text{ and } x \geq 0\} \subset X$.

b) $X = C([-1, 1], \mathbb{R})$ with the metric $d_{\sup}(f, g) = \sup\{|f(x) - g(x)| \mid x \in [-1, 1]\}$ and $U = \{f \in C([-1, 1], \mathbb{R}) \mid f(0) = 0\}$.

Exercise 7 (Continuous functions with a different metric).
Let $C([0, 1], \mathbb{R})$ be the set of all real-valued functions on the interval $[0, 1] \subset \mathbb{R}$. Let

$$d_1(f, g) = \int_0^1 |f(x) - g(x)|\, dx$$

Show that $(C([0, 1], \mathbb{R}), d_1)$ is a metric space.

Challenges

Exercise 8 (The p-adic metric on $\mathbb{Z}$).
Let $p \in \mathbb{N}$ be a prime number. Define a map $d\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ as follows: For $m, n \in \mathbb{Z}$ let $d(m, n) = 0$ if $m = n$, otherwise let $d(m, n) = \frac{1}{r+1}$ where $r \in \mathbb{N}_0$ is chosen in such a way that p^r divides $m - n$, but p^{r+1} does not. Show that d is a metric on $\mathbb{Z}$.

Exercise 9 (The counting metric).
Let E be a finite set and let $X = \mathcal{P}(E)$ be its power set, i.e. the collection of all subsets of E. For $A \in X$ write $\operatorname{card}(A)$ for the number of elements in A (this is also called the cardinality of A). Let

$$d\colon X \times X \to \mathbb{R} \quad ; \quad (A, B) \mapsto \operatorname{card}((A \cup B) \setminus (A \cap B))$$

Show that (X, d) is a metric space!

Hint: The function $I_A\colon X \to \mathbb{R}$ with $I_A(x) = 1$ if $x \in A$ and $I_A = 0$ if $x \notin A$ is called the indicator function of A. Try to express $d(A, B)$ in terms of the indicator functions of A and B.

Exercise 10 (Two metrics on continuous functions).
Let $(C([0,1], \mathbb{R}), d_1)$ be the metric space from Exercise 7. Let

$$d_{\text{sup}}(f, g) = \sup\{|f(x) - g(x)| \mid x \in [0,1]\} \ .$$

Let $0 \in C([0,1], \mathbb{R})$ be the zero function and consider

$$U = \{f \in C([0,1], \mathbb{R}) \mid d_{\text{sup}}(f, 0) < 1\} \ .$$

Show that $U \subset C([0,1], \mathbb{R})$ is open with respect to the metric d_{sup}, but not with respect to the metric d_1.

Hint: Construct a function with maximum $1 + r$ at 0 with area covered less than r.