Problem sheet for the lecture

MA3008 – Algebraic Topology

Sheet 2 Spring Semester 2020

Exam Style Questions

Exercise 1 (Properties of closed sets).

Let $(X, \mathcal{T}_X)$ be a topological space. Show the following facts about closed subsets of X:

a) The empty set $\emptyset$ and the whole space X are closed.

b) If $A \subset X$ and $B \subset X$ are closed, so is their union $A \cup B$.

c) Let I be a set and let $(A_i)_{i \in I}$ be a family of closed subsets of X. Then their intersection $\bigcap_{i \in I} A_i$ is closed as well.

Hint: Use De Morgan's laws.

Exercise 2 (The discrete topology).

Let X be a set and let $d\colon X \times X \to \mathbb{R}$ be the metric on X defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \ , \\ 0 & \text{if } x = y \ . \end{cases}$$

Show that the metric topology $\mathcal{T}(d)$ agrees with the discrete topology $\mathcal{T}_{\text{dis}}$ on X (see Example 2.2.10 in the lecture notes) for this particular choice of metric.

Exercise 3 (Convergent sequences).

Let (X, d) be a metric space. Denote the metric topology by $\mathcal{T}(d)$. Show that a sequence $(a_n)_{n \in \mathbb{N}}$ of points $a_n \in X$ converges to $a \in X$ in the topological space $(X, \mathcal{T}(d))$ if and only if for each $\epsilon > 0$ there is an $N \in \mathbb{N}$ with the property that $d(a_n, a) < \epsilon$ for all $n > N$.

Exercise 4 (Basis).

Let X be a set and let $\mathcal{B} = \{X\}$ be the family that contains only the element X.

a) Check that this family $\mathcal{B}$ is the basis of a topology $\mathcal{T}_{\mathcal{B}}$ on X.

b) We already know this topology. Which topology $\mathcal{T}_{\mathcal{B}}$ on X do you get this way?

Intermediate Level Questions

Exercise 5 (The indiscrete topology).
Let X be a set and consider the topological space $(X, \mathcal{T}_{\text{ind}})$, where $\mathcal{T}_{\text{ind}}$ is the indiscrete topology. Let (Y, d_Y) be a metric space and consider it as a topological space equipped with the metric topology $\mathcal{T}(d_Y)$.

a) Let $y \in Y$. Show that $U = Y \setminus \{y\} \subset Y$ is open with respect to $\mathcal{T}(d_Y)$.

b) Show that any continuous map $f\colon X \to Y$ has to be constant.

Exercise 6 (Intersection of topologies).
Let X and I be sets and let $(\mathcal{T}_i)_{i\in I}$ be a family of topologies on X indexed by I. Prove that $\mathcal{T} = \bigcap_{i\in I} \mathcal{T}_i$ is a topology on X.

Exercise 7 (Subspace topology on $\mathbb{Z}$).
Let $\mathbb{R}$ be the real line equipped with the metric $d\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ given by $d(x,y) = |x-y|$. Consider the topological space $(\mathbb{R}, \mathcal{T}(d))$ and let $\mathbb{Z} \subset \mathbb{R}$ be the subset of integers. Show that the subspace topology $\mathcal{T}_{\mathbb{Z}\subset\mathbb{R}}$ on $\mathbb{Z}$ induced by $\mathbb{R}$ agrees with the discrete topology $\mathcal{T}_{\text{dis}}$ on $\mathbb{Z}$.

Challenges

Exercise 8 (Convergent sequences as continuous maps).
Consider the set $\mathbb{N}_+ = \mathbb{N} \cup \{\infty\}$, i.e. the set of all natural numbers plus an additional point, which we call ∞. We define a topology $\mathcal{T}_+$ on $\mathbb{N}_+$ in the following way: All subsets of $\mathbb{N}$ are in $\mathcal{T}_+$ and all sets that contain ∞ and a complement of a finite set are in $\mathcal{T}_+$. For example, the set $\{n \in \mathbb{N} \mid n > 5\} \cup \{\infty\}$ is in $\mathcal{T}_+$, while the sets $\{2k \mid k \in \mathbb{N}\} \cup \{\infty\}$ and $\{1, 2, 3, \infty\}$ are not.

a) Show that $\mathcal{T}_+$ is in fact a topology on $\mathbb{N}_+$.

b) Let $\mathbb{N}$ be equipped with the discrete topology. Show that the inclusion $\iota\colon \mathbb{N} \to \mathbb{N}_+$ is continuous.

c) Let $(Y, \mathcal{T}_Y)$ be another topological space and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in Y. It gives rise to a continuous map $f\colon \mathbb{N} \to Y$ with $f(n) = a_n$. Show that the sequence converges if and only if there is a continuous map $F\colon \mathbb{N}_+ \to Y$, such that $F \circ \iota = f$.

The space $\mathbb{N}_+$ is also called the *one-point compactification of* $\mathbb{N}$.

Exercise 9 (Proof of Euclid's theorem via topology).
The goal of this exercise is to prove Euclid's theorem, which states that there are infinitely many prime numbers. The following surprising *topological* proof was discovered in 1955 by H. Fürstenberg. Enjoy!

Recall that a natural number $p \in \mathbb{N}$ is prime if $p \neq 1$ and if its only divisors are 1 and p. Let $\mathbb{P} \subset \mathbb{N}$ be the subset of all prime numbers.

a) Show that every natural number $n \neq 1$ is divisible by a prime number.

b) For $x, n \in \mathbb{Z}$ define
$$x + n\mathbb{Z} = \{x + nz \mid z \in \mathbb{Z}\} .$$
Call a subset $U \subset \mathbb{Z}$ open iff for every $x \in U$ there exists an $n \in \mathbb{N} \setminus \{0\}$ such that $x + n\mathbb{Z} \subset U$. Show that the set of all these open subsets $U \subset \mathbb{Z}$ defines a topology on $\mathbb{Z}$.

c) Show that for every $n \in \mathbb{N} \setminus \{0\}$, the subset $n\mathbb{Z} = 0 + n\mathbb{Z} \subset \mathbb{Z}$ is both open and closed in $\mathbb{Z}$ with respect to the topology defined in part b).

d) Using part a), prove that
$$\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \in \mathbb{P}} p\mathbb{Z} .$$

e) Conclude that $\mathbb{P}$ must contain infinitely many elements.