Problem sheet for the lecture

MA3008 – Algebraic Topology

Sheet 3 Spring Semester 2020

Exam Style Questions

Exercise 1 (An equivalence relation).
Prove that "being homeomorphic" is an equivalence relation on topological spaces.

Exercise 2 (Subspace topology).
Let $(X, \mathcal{T}_X), (Z, \mathcal{T}_Z)$ be topological spaces and let $Y \subset X$ be a subspace of X (i.e. a subset equipped with the subspace topology). Let $\iota\colon Y \to X$ be the inclusion map and let $f\colon Z \to Y$ be a map. Show that f is continuous if and only if $h = \iota \circ f$ is continuous. A diagram of the maps is shown below:

$$\begin{array}{ccc} Z & \xrightarrow{f} & Y \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle \iota} \\ & & X \end{array}$$

Exercise 3 (Graphs of maps).
Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f\colon X \to Y$ be a map and define the graph Γ_f of f as follows:

$$\Gamma_f = \{(x, y) \in X \times Y \mid y = f(x)\} \subset X \times Y$$

Note that Γ_f is a topological space when it is equipped with the subspace topology induced by the product topology on $X \times Y$. Show that f is continuous if and only if the map $g\colon X \to \Gamma_f$ given by $g(x) = (x, f(x))$ is a homeomorphism.

Hint: You have to use Exercise 2 at some point.

Exercise 4 (Product topology and closed sets).
Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $A \subset X$ and $B \subset Y$ be subsets. Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Intermediate Level Questions

Exercise 5 (Subspace topology, Convergence of sequences).
Let $X = \mathbb{R}$ be equipped with the metric topology from the metric $d(x, y) = |x - y|$. Let

$$Y = \left\{ x \in \mathbb{R} \mid x = \frac{1}{n}, n \in \mathbb{N} \right\} \cup \{0\} \subset \mathbb{R}$$

be equipped with the subspace topology.

a) Let $U \subset Y$ be an open subset with $0 \in U$. Show that there is $y \in U$ with $y \neq 0$. Deduce that the subspace topology $\mathcal{T}_{Y \subset X}$ on Y is not the same as the discrete topology $\mathcal{T}_{\text{dis}}$ on Y.

b) Let $(Z, \mathcal{T}_Z)$ be another topological space and let $f \colon Y \to Z$ be a continuous map. Define $a_n = f(\frac{1}{n})$ for $n \in \mathbb{N}$. Show that the sequence $(a_n)_{n \in \mathbb{N}}$ converges to $a = f(0)$ in Z.

Exercise 6 (Subspace topology and product topology).
Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let A be a subspace of X and let B be a subspace of Y. We equip A and B with the subspace topologies $\mathcal{T}_{A \subset X}$ and $\mathcal{T}_{B \subset Y}$, respectively. Prove that the product topology on $A \times B$ is the same as the topology $\mathcal{T}_{A \times B \subset X \times Y}$, i.e. as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Exercise 7 (Subspace topology and homeomorphisms).
Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f \colon X \to Y$ be a homeomorphism. Let $A \subset X$ be equipped with the subspace topology $\mathcal{T}_{A \subset X}$ and define

$$f(A) = \{ f(x) \in Y \mid x \in A \} \ .$$

This is a topological space equipped with the subspace topology $\mathcal{T}_{f(A) \subset Y}$. Show that $g \colon A \to f(A)$ with $g(x) = f(x)$ for all $x \in A$ and $h \colon X \setminus A \to Y \setminus f(A)$ given by $h(x) = f(x)$ for all $x \in X \setminus A$ are both homeomorphisms.

Hint: You can solve this by constructing a continuous map $h \colon f(A) \to A$ that is inverse to g. Exercise 2 will come in handy again when continuity needs to be checked.

Challenges

Exercise 8 (Infinite products).
Let I be a set and let $(X_i, \mathcal{T}_i)_{i\in I}$ be a family of topological spaces indexed by I. Define the product $\prod_{i\in I} X_i$ as follows

$$\prod_{i\in I} X_i = \left\{ f\colon I \to \bigcup_{i\in I} X_i \mid f(i) \in X_i \right\} ,$$

For $f \in \prod_{i\in I} X_i$ we will denote the value $f(i) \in X_i$ by f_i. Let $(U_i)_{i\in I}$ be a family of subsets with $U_i \subset X_i$ for all $i \in I$. Define $\prod_{i\in I} U_i$ to be the subset containing all $f \in \prod_{i\in I} X_i$ with $f_i \in U_i$ for all $i \in I$. Consider the following family:

$$\mathcal{B} = \left\{ \prod_{i\in I} U_i \subset \prod_{i\in I} X_i \mid U_i \in \mathcal{T}_i \text{ for all } i \in I \text{ and } U_i \neq X_i \text{ for only finitely many } i \in I \right\}$$

a) Show that $\mathcal{B}$ is the basis of a topology $\mathcal{T}$ on $\prod_{i\in I} X_i$. (This topology is called the product topology on $\prod_{i\in I} X_i$.)

b) Prove that the projection maps $p_j\colon \prod_{i\in I} X_i \to X_j$ for all $j \in I$ with $p_j(f) = f_j$ are continuous if the domain is equipped with the topology $\mathcal{T}$ from part a) and X_j is equipped with the topology $\mathcal{T}_j$.

c) Now consider the following family of subsets

$$\mathcal{B}_{\text{box}} = \left\{ \prod_{i\in I} U_i \subset \prod_{i\in I} X_i \mid U_i \in \mathcal{T}_i \text{ for all } i \in I \right\} .$$

Show that $\mathcal{B}_{\text{box}}$ is also the basis of a topology $\mathcal{T}_{\text{box}}$ on $\prod_{i\in I} X_i$. Prove that all p_j from part b) are still continuous if the domain is equipped with this topology. (The topology $\mathcal{T}_{\text{box}}$ is called the box topology on $\prod_{i\in I} X_i$ and if I is finite, it agrees with the product topology.)

d) "Wait a minute!", I hear you say, "The topology in part c) looks like the 'right' topology on $\prod_{i\in I} X_i$. Why is that one not called the product topology?" Well, it is inconvenient in many situations, which can be illustrated as follows: Let $I = \mathbb{N}$ and let $X_i = \mathbb{R}$ for all $i \in \mathbb{N}$, where $\mathbb{R}$ is equipped with its metric topology $\mathcal{T}(d)$ from $d(x, y) = |x - y|$. Show that the map

$$f\colon \mathbb{R} \to \prod_{i\in\mathbb{N}} \mathbb{R} \quad , \quad x \mapsto (x, x, x, \dots)$$

is **not** continuous if the codomain is equipped with the *box topology* and that it **is** continuous if we use the *product topology*.