Problem sheet for the lecture

MA3008 – Algebraic Topology

Sheet 4 Spring Semester 2020

Exam Style Questions

Exercise 1 (Connectedness).
Let $\mathbb{R}$ be equipped with its usual metric topology and let $S^0 = \{-1, 1\} \subset \mathbb{R}$. Prove that the subspace $\mathbb{R} \setminus S^0$ is disconnected.

Bonus Question: Is $\mathbb{R}^{n+1} \setminus S^n$ for $n \in \mathbb{N}$ disconnected as well?

Exercise 2 (Path-connectedness).
Let $\mathbb{R}^2$ be equipped with its usual metric topology and let $S^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\} \subset \mathbb{R}^2$ be the circle considered as a subspace of $\mathbb{R}^2$. Let $x_0 \in S^1$. Show that the subspace

$$X = S^1 \setminus \{x_0\} \subset S^1$$

is path-connected.

Exercise 3 (Connectedness and homeomorphisms).
Let $\mathbb{R}^n$ for $n \in \mathbb{N}$ be equipped with its usual metric topology. Let $S^1 \subset \mathbb{R}^2$ be the circle as a subspace of $\mathbb{R}^2$ (see Exercise 2). Show that the two topological spaces S^1 and $\mathbb{R}$ are **not** homeomorphic.

Exercise 4 (Connectedness and continuous maps).
Let $\mathbb{R}$ and $\mathbb{R}^2$ be equipped with their usual metric topologies and consider the subspacess $S^1 \subset \mathbb{R}^2$ and $S^0 = \{-1, 1\} \subset \mathbb{R}$.

a) Let $f\colon S^1 \to S^0$ be a continuous map. Prove that it has to be equal either to the constant map with value 1 or the constant map with value -1.

b) Suppose that $f\colon S^1 \to \mathbb{R}$ is a continuous map. Prove that there has to be a point $x \in S^1$, such that $f(x) = f(-x)$.

   Hint: Suppose that there is no such point and consider the map

$$g\colon S^1 \to S^0 \quad ; \quad g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|} \; .$$

Intermediate Level Questions

Exercise 5 (Quotient spaces).

Define an equivalence relation on $\mathbb{R}$ by $x \sim y$ if and only if $x = y = 0$ or $xy > 0$. Consider $\mathbb{R}$ as a topological space equipped with the metric topology from $d(x, y) = |x - y|$.

a) How many elements are there in $\mathbb{R}/\!\sim$?

b) Write down all sets in the quotient topology on $\mathbb{R}/\!\sim$, i.e. all open subsets of $\mathbb{R}/\!\sim$.

c) Is $\mathbb{R}/\!\sim$ path-connected?

d) Is $\mathbb{R}/\!\sim$ compact?

Exercise 6 (Connectedness and quotient spaces).

Let $(X, \mathcal{T}_X)$ be a topological space and let $\sim$ be an equivalence relation on X. Let $X/\!\sim$ be equipped with the quotient topology $\mathcal{T}_{X/\sim}$. Suppose $X/\!\sim$ is connected and that each equivalence class $[x] \in X/\!\sim$ is connected when considered as a subspace of X. Show that X is connected.

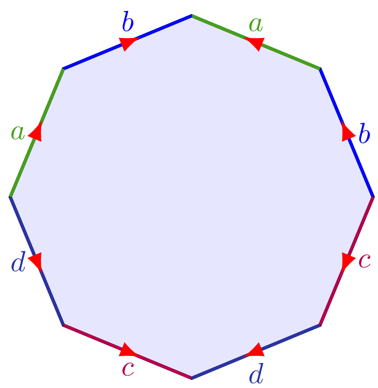
Exercise 7 (A weird quotient space).

Let $\mathbb{R}$ be equipped with the usual metric topology. Define an equivalence relation on $\mathbb{R}$ by $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Prove that the topology on $\mathbb{R}/\!\sim$ agrees with the indiscrete topology.

Challenges

Exercise 8 (A nice quotient space).

Consider the octagon Y shown below as a subspace of $\mathbb{R}^2$. Draw the topological space X obtained as the quotient of Y with edges identified as in the diagram below. (X can be nicely embedded in $\mathbb{R}^3$!)



Exercise 9 (A space that is not locally connected).

A topological space $(X, \mathcal{T}_X)$ is called **locally connected at** $x \in X$, if every neighbourhood of x contains a connected open neighbourhood of x. If X is locally connected at all of its points, then we call it locally connected. Consider the following subspace

$$Y = \{(x, 0) \mid x \in [0,1]\} \,\cup\, \bigcup_{n \in \mathbb{N}} \left\{ \left(t, \frac{t}{n} \right) \mid t \in [0,1] \right\} \subset \mathbb{R}^2$$

a) The space Y is called the “infinite broom”. Draw a sketch of Y to see why.

b) Show that Y is path-connected.

c) Show that Y is not locally connected.