Problem sheet for the lecture

MA3008 – Algebraic Topology

Sheet 5 Spring Semester 2020

Exam Style Questions

Exercise 1 (Compactness).
In this exercise $\mathbb{R}^n$ for $n \in \mathbb{N}$ will be considered as a topological space equipped with its standard metric topology. Which of the following topological spaces X are compact?

a) $X = [a, b) \subset \mathbb{R}$ for $a, b \in \mathbb{R}$ with $a < b$, where X is equipped with the subspace topology,

b) $X = [0, 1]/\sim$, where $[0, 1] \subset \mathbb{R}$ is equipped with the subspace topology, X with the quotient topology and $\sim$ is the equivalence relation given by

$$x \sim y \quad \text{if and only if} \quad (x = y \text{ or } x, y \in \{0, 1\}) \ ,$$

c) $X = [0, 1] \cup [3, 4] \subset \mathbb{R}$, where X is equipped with the subspace topology,

d) any closed subset $X \subset D^n$, where $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ is equipped with subspace topology.

Exercise 2 (Intersections of compact spaces).
Let X be a Hausdorff space. Let $C_1 \subset X$ and $C_2 \subset X$ be compact subspaces. Show that $C_1 \cap C_2$ is compact.

Exercise 3 (Some examples of groups).
Which of the following sets G forms a group with respect to the given multiplication operations?

a) $G = \mathbb{Q}_+$, the set of all positive rational numbers, with respect to multiplication,

b) $G = \mathbb{Z}$ with respect to multiplication,

c) $G = \{0, 1, 2\}$ with the operation given by adding the two numbers and then taking their remainder with respect to division by 3 (e.g. $2 + 2 = 1$ and $1 + 2 = 0$).

Exercise 4 (Fundamental groups).
In this exercise $\mathbb{R}^n$ for $n \in \mathbb{N}$ will be considered as a topological space equipped with its standard metric topology. Let $S^1 \subset \mathbb{R}^2$ be the circle, let $I = [0, 1] \subset \mathbb{R}$ be the unit interval. Consider the product space $X = S^1 \times \mathbb{R} \times I$ and let $x_0 = (z_0, 0, 1) \in X$. Use theorems from the lectures and Exercise 8 to prove that $\pi_1(X, x_0) \cong \mathbb{Z}$.

Intermediate Level Questions

Exercise 5 (Compactness and infinite intersections).
Let X be a compact topological space and let $(F_n)_{n\in\mathbb{N}}$ be a sequence of non-empty closed subsets $F_n \subset X$ such that $F_{n+1} \subset F_n$ for all $n \in \mathbb{N}$. Prove that

$$\bigcap_{n\in\mathbb{N}} F_n \neq \emptyset \, .$$

Exercise 6 (Homotopic maps and contractible spaces).
Let $(X, \mathcal{T}_X)$ be a contractible topological space and let $(Y, \mathcal{T}_Y)$ be a path-connected topological space. Prove that all continuous maps $f\colon X \to Y$ are homotopic to one another.

Exercise 7 (Homotopic maps and fundamental group).
Consider $\mathbb{R}^n$ as a topological space equipped with the standard metric topology. A subspace $A \subset \mathbb{R}^n$ is called ***convex*** if for all $x, y \in A$ and all $t \in [0,1]$ we have $tx + (1-t)y \in A$, i.e. the straight line between any two points of A runs entirely through A.

a) Consider a convex subspace $A \subset \mathbb{R}^n$ and suppose $A \neq \emptyset$ and let $a \in A$. Prove that the constant map $c_a\colon A \to A$ given by $c_a(x) = a$ is homotopic to the identity map $\mathrm{id}_A\colon A \to A$ with $\mathrm{id}_A(x) = x$.
b) Let A and $a \in A$ be as in a). Show that the homotopy H between c_a and id_A can be chosen to be based with base point $a \in A$.
c) Let A and $a \in A$ be as in a). Deduce that $\pi_1(A, a)$ just contains one element.

Exercise 8 (Fundamental group of product spaces).
Let (X, x_0) and (Y, y_0) be pointed topological spaces and denote by $p_X\colon X \times Y \to X$ and $p_Y\colon X \times Y \to Y$ the projection maps. Let $\gamma\colon I \to X \times Y$ be a loop in $X \times Y$ at (x_0, y_0). Show that

$$\theta\colon \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

with $\theta([\gamma]_+) = ([p_X \circ \gamma]_+, [p_Y \circ \gamma]_+)$ is a well-defined group isomorphism.

Challenges

Exercise 9 (The fundamental group of topological groups).

A ***topological group*** is a topological space $(G, \mathcal{T}_G)$ which is also a group in such a way that the group multiplication

$$\cdot : G \times G \to G \quad , \quad (g_1, g_2) \mapsto g_1 \cdot g_2$$

and the map $i\colon G \to G$ given by $i(g) = g^{-1}$ are both continuous. For example, the space of all orthogonal matrices, i.e. $O(n) = \{A \in M_{n\times n}(\mathbb{R}) \mid AA^t = A^tA = 1_n\}$ is a subspace of $\mathbb{R}^{n^2}$ and a group with respect to matrix multiplication. Since the group multiplication and taking inverses are both continuous, $O(n)$ is a topological group. In this exercise we will show that $\pi_1(G, e)$ (where $e \in G$ denotes the neutral element) is an abelian group for any topological group G.

a) Suppose that X is a set equipped with two multiplications, denoted by $\cdot$ and $*$, such that there is an element $1 \in X$, which is a neutral element for both multiplications, i.e.

$$1 \cdot x = x \cdot 1 = x \quad , \quad 1 * x = x * 1 = x$$

for all $x \in X$. Moreover, assume that the following equality holds

$$(a * b) \cdot (c * d) = (a \cdot c) * (b \cdot d)$$

for all $a, b, c, d \in X$. Prove that $*$ and $\cdot$ agree and are both abelian and associative.

b) Let $\gamma_1, \gamma_2, \gamma_1', \gamma_2'\colon I \to G$ be loops in G starting and ending at the neutral element $e \in G$. Suppose that $\gamma_i \sim_{h,+} \gamma_i'$ for $i \in \{1, 2\}$. Define $\gamma_3\colon I \to G$ by $\gamma_3(t) = \gamma_1(t) \cdot \gamma_2(t)$ (i.e. $\gamma_3 = \gamma_1 \cdot \gamma_2$) and define γ_3' similarly using γ_1' and γ_2'. Show that

$$\gamma_3 \sim_{h,+} \gamma_3'$$

and deduce that this induces another multiplication $\pi_1(G, e) \times \pi_1(G, e) \to \pi_1(G, e)$ given by $[\gamma_1]_+ \cdot [\gamma_2]_+ = [\gamma_1 \cdot \gamma_2]_+$.

c) Let $\cdot$ be the multiplication from part b) and denote by $*$ the usual multiplication in $X = \pi_1(G, e)$ induced by concatenation of loops. Show that these two multiplications satisfy the conditions in part a). Deduce that $\pi_1(G, e)$ is an abelian group.