Bundles of C^*-Algebras

An Introduction to Dixmier-Douady Theory

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13th August 2020

UK Virtual Operator Algebras Seminar

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What is a bundle? - an analytic perspective

There are (at least) two different notions of bundle:

Let X be a locally compact Hausdorff space.

Definition

A C^*-algebra A is a **$C_0(X)$-algebra** if it comes equipped with a non-degenerate $*$-homomorphism

$$\theta \colon C_0(X) \to Z(M(A)) \,.$$

A is a **continuous $C_0(X)$-algebra** if the function $x \mapsto \|a(x)\|$ is continuous for all $a \in A$.

- given closed $Y \subset X$, let $A(Y) = A / C_0(X \setminus Y) \cdot A$,
- fibre of A at x is $A(x) := A(\{x\})$,
- let $a \in A$, denote the image of a in $A(x)$ by $a(x)$.

Examples of $C_0(X)$-algebras

- the **trivial $C_0(X)$-algebra** $C_0(X, B)$ with fibre B, ie. C_0-maps $f : X \to B$ for a C^*-algebra B, is a continuous $C_0(X)$-algebra,
- Let B be a unital C^*-algebra and consider

  $$A = \{f \in C([-1, 1], B) \mid f(0) \in \mathbb{C}\, 1_B\} \ .$$

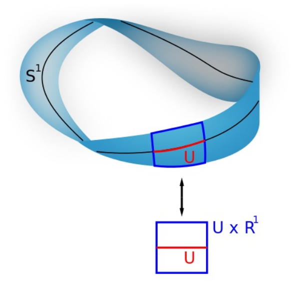
  This is also a continuous $C_0(X)$-algebra.
- Suppose we have a central extension of discrete countable amenable groups

  $$1 \to N \to G \to H \to 1 \, .$$

  Then $C^*(G)$ is a continuous $C(\hat{N})$-algebra, where $\hat{N}$ is the Pontrjagin dual of N. The fibre over the trivial char. is $C^*(H)$.

What is a bundle? - a topological perspective

When topologists hear the word “bundle”, they think of something like this...



- locally a product (ie. locally trivial),
- but (possibly) non-trivial globally.

What is a bundle? - a topological perspective

Definition

A (locally trivial) **fibre bundle** $E \to X$ with **fibre** F consists of

- a topological space E, called the **total space**,
- a continuous map $\pi\colon E \to X$

with the following property:

- each point $x \in X$ has an open neighbourhood $U \subset X$ such that there exists a homeomorphism $\varphi_U\colon U \times F \to E|_U$ that makes

  $$\begin{array}{ccc} U \times F & \xrightarrow{\varphi_U} & E|_U \\ & {\scriptstyle \mathrm{pr}_U} \searrow \quad \swarrow {\scriptstyle \pi|_U} & \\ & U & \end{array}$$

  commute. The map φ_U is called a **trivialisation** over U.

What is a bundle? - a topological perspective

Let $E \to X$ be a fibre bundle with fibre F.

- Take an open cover $(U_i)_{i \in I}$ with trivialisations φ_i over U_i.
- The **transition maps** φ_{ij} over $U_{ij} = U_i \cap U_j$ are

$$\varphi_{ij} \colon U_{ij} \to \mathrm{Homeo}(F) \; .$$

  with $\varphi_{ij}(x)(f) = (\varphi_j^{-1} \circ \varphi_i)(x, f)$.
- Let $G \subset \mathrm{Homeo}(F)$. The bundle E has **structure group** G if we can find an open cover $(U_i)_{i \in I}$ such that all φ_{ij} factor through G.

Examples of fibre bundles

- vector bundles over a topological field k are fibre bundles with structure group $GL_n(k)$,
- hermitian vector bundles are fibre bundles with structure group $U(n)$
- manifold bundles with fibre a closed smooth manifold M are fibre bundles with structure group $\mathrm{Diff}(M)$,

And finally...

Definition

Let B be a C^*-algebra. A **bundle of C^*-algebras** $\mathcal{A} \to X$ with fibre B is a fibre bundle with structure group $\mathrm{Aut}(B)$ (equipped with the point-norm topology).

Homotopy classification of fibre bundles

Theorem

Let G be a topological group. There exists a topological space BG (called ***classifying space*** *of G) and a fibre bundle*

$$EG \to BG$$

with structure group G that has the following property:

For every compact Hausdorff space X and every fibre bundle $E \to X$ with structure group G there exists a continuous map $f : X \to BG$ (unique up to homotopy) such that

$$\begin{array}{ccc} E & \longrightarrow & EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BG \end{array}$$

is a pullback diagram.

In other words...

Corollary

Let $\mathcal{B}un_G(X)$ be the set of isomorphism classes of fibre bundles with structure group G. There is a natural $1:1$-correspondence

$$\mathcal{B}un_G(X) \leftrightarrow [X, BG]$$

Examples of classifying spaces:

- $B\mathbb{Z} \simeq S^1$, hence $[X, B\mathbb{Z}] \cong [X, S^1] \cong H^1(X, \mathbb{Z})$.
- Hermitian line bundles have structure group $U(1)$ and

$$[X, BU(1)] \cong H^2(X, \mathbb{Z}) \ .$$

Back to $C_0(X)$-algebras

Let $\pi\colon \mathcal{A} \to X$ be a bundle of C^*-algebras with fibre B and consider

$$C_0(X, \mathcal{A}) = \{f\colon X \to \mathcal{A} \mid f \text{ is a } C_0\text{-map and } \pi \circ f = \mathrm{id}_X\}$$

This **section algebra** is a continuous $C_0(X)$-algebra.

Question

Given a $C_0(X)$-algebra A. When is it locally trivial?

1. Let $M_n = M_n(\mathbb{C})$ and consider

$$A = \{f \in C([-1,1], M_2) \mid f(0) \in \mathbb{C}\,1_2\}$$

Is this isomorphic (as a $C([-1,1])$-algebra) to the trivial one?

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No! Fibre of A over 0 is $\cong \mathbb{C} \not\cong M_2$.

Examples

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   $$1_2 \otimes e_{11} \in \mathbb{C}1_2 \otimes \mathbb{K}$$

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Observation

There is **no** continuous path in $\hom(\mathbb{K}, M_2 \otimes \mathbb{K})$ connecting $T \mapsto 1_2 \otimes T$ to an isomorphism.

Continuous $C_0(X)$-algebras with fibre $\mathbb{K}$

Reminder: $\mathbb{K} = \mathbb{K}(H)$ with H separable, ∞-dimensional.

Definition

Let A be a continuous $C_0(X)$-algebra with all fibres isomorphic to $\mathbb{K}$. It is said to satisfy **Fell's condition** if

$$\forall x \in X\, \exists \text{ closed nb. } V \text{ and } p \in A(V) \text{ s.th. } p(y) \text{ has rank } 1\, \forall y \in V\ .$$

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Theorem (Fell)

Let A be a sep. continuous $C_0(X)$-algebra with fibre $\mathbb{K}$ over a locally compact, second countable space X of fin. dim. Then

$$A \text{ locally trivial} \quad \Leftrightarrow \quad A \text{ satisfies Fell's condition} \ .$$

Continuous $C_0(X)$-algebras with fibre $\mathbb{K}$

What about the homotopy classification?

Note that all $*$-automorphisms of $\mathbb{K}$ are inner, ie.

$$\mathrm{Aut}(\mathbb{K}) \cong PU(H) := U(H)/U(1) \ .$$

Need to understand homotopy type of $BPU(H)$.

- $\pi_k(BPU(H)) \cong \pi_{k-1}(PU(H))$ (by classifying space theory),
- $\dots \to \pi_k(U(H)) \to \pi_k(PU(H)) \to \pi_{k-1}(U(1)) \to \pi_{k-1}(U(H)) \to \dots$
- but $U(H)$ contractible

$$\pi_k(B\mathrm{Aut}(\mathbb{K})) \cong \begin{cases} \mathbb{Z} & k = 3 \\ 0 & \text{else} \end{cases}$$

Definition

Let G be a discrete group (abelian if $n \geq 2$). A space $K(G, n)$ with

$$\pi_k(K(G,n)) \cong \begin{cases} G & k = n \\ 0 & \text{else} \end{cases}$$

is called an **Eilenberg-Maclane space**. It is unique up to homotopy equivalence.

Theorem (Eilenberg-Steenrod)

Let X be a compact space. Then we have a natural isomorphism

$$[X, K(\mathbb{Z}, n)] \cong \check{H}^n(X, \mathbb{Z}) \ .$$

Continuous $C_0(X)$-algebras with fibre $\mathbb{K}$

Since $BAut(\mathbb{K}) \simeq K(\mathbb{Z}, 3)$ we have

Corollary (Dixmier-Douady)

Let X be a compact space. Then we have a natural isomorphism:

$$\delta \colon \mathcal{B}un_{\mathbb{K}}(X) \to \check{H}^3(X, \mathbb{Z})$$

called the ***Dixmier-Douady class****. This isomorphism is multiplicative in the sense that*

$$\delta(A_1 \otimes_{C(X)} A_2) = \delta(A_1) + \delta(A_2)\ .$$

Let A^{op} be the continuous $C(X)$-algebra with reversed multiplication, then

$$\delta(A^{op}) = -\delta(A)\ .$$

Generalised Dixmier-Douady Theory - An Example

3. Remember our example from the beginning:

$$A = \{f \in C([-1,1], M_2) \mid f(0) \in \mathbb{C}\,1_2\} \ .$$

Consider $A'' = A \otimes M_{2^\infty} \otimes \mathbb{K}$ with $M_{2^\infty} = M_2^{\otimes \infty}$.
Is this trivialisable?

Generalised Dixmier-Douady Theory - An Example

3 Remember our example from the beginning:

$$A = \{f \in C([-1,1], M_2) \mid f(0) \in \mathbb{C}\,1_2\} \ .$$

Consider $A'' = A \otimes M_{2^\infty} \otimes \mathbb{K}$ with $M_{2^\infty} = M_2^{\otimes\infty}$.
Is this trivialisable?

Yes! We can find a continuous path

$$\gamma \colon [0,1] \to \mathbf{hom}(M_{2^\infty} \otimes \mathbb{K}, M_2 \otimes M_{2^\infty} \otimes \mathbb{K})$$

with the properties

- $\gamma(0)(T) = 1_2 \otimes T$,
- $\gamma(t)$ is an isomorphism for each $t \in (0,1]$.

This gives us an isomorphism

$$C([-1,1], M_{2^\infty} \otimes \mathbb{K}) \to A''$$

by applying $\gamma(|t|)$ to $f(t)$ for $t \in [-1,1]$.

Generalised Dixmier-Douady Theory

This example works, since $D = M_{2^\infty}$ is strongly self-absorbing!

Definition (Toms-Winter)

A separable, unital C^*-algebra D is called **strongly self-absorbing** if $\exists$ an isomorphism $\psi \colon D \to D \otimes D$ and a path $u \colon [0,1) \to U(D \otimes D)$ with

$$\lim_{t \to 1} \|\psi(d) - u_t(d \otimes 1_D)u_t^*\| \to 0$$

Some consequences of this definition:

- $\mathrm{Aut}(D)$ is contractible (and so is $B\mathrm{Aut}(D)$),
- $K_0(D)$ is a ring (and $K_1(D) = 0$ if D satisfies the UCT),

Definition (Dadarlat-P.)

A continuous $C_0(X)$-algebra A with fibre $D \otimes \mathbb{K}$ for a strongly self-abs. C^*-algebra D satisfies the **generalised Fell condition** if

$$\forall x \in X \, \exists \text{ closed nb. } V \text{ and } p \in A(V) : [p(y)] \in GL_1(K_0(A(y))) \; \forall y \in V \, .$$

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Theorem (Dadarlat-P.)

Let X be a locally compact space of finite covering dimension, let A be a separable $C_0(X)$-algebra as in the definition. Then

$$A \text{ locally trivial} \quad \Leftrightarrow \quad A \text{ satisfies the generalised Fell condition} \, .$$

C^*-algebra bundles with fibre $D \otimes \mathbb{K}$

What about the homotopy classification?

Theorem (Dadarlat-P.)

$\mathcal{B}un_{D\otimes\mathbb{K}}(X)$ is a group with respect to $\otimes_X$. In particular,

$$\mathcal{B}un_{M_{\mathbb{Q}}\otimes\mathbb{K}}(X) \cong H^1(X, \mathbb{Q}_+^\times) \oplus H^{odd,\geq 3}(X, \mathbb{Q}) \ ,$$
$$\mathcal{B}un_{M_{\mathbb{Q}}\otimes\mathcal{O}_\infty\otimes\mathbb{K}}(X) \cong H^1(X, \mathbb{Q}^\times) \oplus H^{odd,\geq 3}(X, \mathbb{Q})$$

If D is strongly self-absorbing, satisfies the UCT and $K_0(D) \neq 0$, then

$$D \otimes M_{\mathbb{Q}} \otimes \mathcal{O}_\infty \cong M_{\mathbb{Q}} \otimes \mathcal{O}_\infty \ .$$

This induces a homomorphism

$$\delta \colon \mathcal{B}un_{D\otimes\mathbb{K}}(X) \to H^1(X, \mathbb{Q}^\times) \oplus H^{\mathrm{odd},\geq 3}(X, \mathbb{Q}) \ .$$

(in particular: $\delta(A_1 \otimes_{C(X)} A_2) = \delta(A_1) + \delta(A_2)$).

C^*-algebra bundles with fibre $\mathcal{O}_\infty \otimes \mathbb{K}$

Reminder: $PU(H) \simeq K(\mathbb{Z}, 2)$

object	classifying space
hermitian line bundle L	$PU(H) \simeq BU(1)$
bundle of compact operators $\mathcal{A}$	$BPU(H) \simeq BBU(1)$

$$U(1) \text{ is an abelian group} \quad \Rightarrow \quad BU(1), BBU(1), \dots \text{ exist}$$

Observation: Line bundles form a subgroup in $GL_1(K^0(X))$.
Idea: Extend the above table to all of $GL_1(K^0(X))$!

object	classifying space
virtual line bundles	$GL_1(KU)$
?	$BGL_1(KU)$

$$GL_1(KU) \text{ is an infinite loop space} \quad \Rightarrow \quad BGL_1(KU) \text{ exists}$$

C^*-algebra bundles with fibre $\mathcal{O}_\infty \otimes \mathbb{K}$

Theorem (Dadarlat-P.)

In general $\mathcal{B}un_{D\otimes\mathcal{O}_\infty\otimes\mathbb{K}}(X)$ is isomorphic to the first group of the cohomology theory associated to the unit spectrum of K-theory with coefficients $K_0(D)$. In particular, we have group isomorphisms:

$$\mathcal{B}un_{\mathcal{O}_\infty\otimes\mathbb{K}}(X) \cong [X, BGL_1(KU)] \ ,$$
$$\mathcal{B}un_{M_{n^\infty}\otimes\mathcal{O}_\infty\otimes\mathbb{K}}(X) \cong [X, BGL_1(KU[\tfrac{1}{d}])] \ .$$

Remarks:

- We also determined the homotopy type of $B\mathrm{Aut}(D \otimes \mathbb{K})$ for all strongly self-absorbing C^*-algebras D,
- The group $[X, BGL_1(KU)]$ and its variants can be determined via the Atiyah-Hirzebruch spectral sequence.

What is next?

- Is there a similar classification for bundles with group actions using equivariant stable homotopy theory?
- Fell bundles with unit fibre D give interesting examples. Can we compute the Dixmier-Douady classes?
- Applications to (equivariant) higher twists in K-theory (Freed-Hopkins-Teleman, T-duality, ...)
- Is there a version of Chern-Weil theory to compute the generalised rational Dixmier-Douady classes?
- Can we use our results to say something about bundles of C^*-algebras with fibre $\mathcal{O}_n$? (work of Taro Sogabe)

Thank you!